

Characterizations of SLE_κ for $\kappa \in (4, 8)$ on Liouville quantum gravity

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Abstract

We prove that SLE_κ for $\kappa \in (4, 8)$ on an independent $\gamma = 4/\sqrt{\kappa}$ -Liouville quantum gravity (LQG) surface is uniquely characterized by the form of its LQG boundary length process and the form of the conditional law of the unexplored quantum surface given the explored curve-decorated quantum surface up to each time t . We prove variants of this characterization for both whole-plane space-filling SLE_κ on a γ -quantum cone (which is the setting of the peanosphere construction) and for chordal SLE_κ on a single bead of a $\frac{3\gamma}{2}$ -quantum wedge. Using the equivalence of Brownian and $\sqrt{8/3}$ -LQG surfaces, we deduce that SLE_6 on the Brownian disk is uniquely characterized by the form of its boundary length process and that the complementary connected components of the curve up to each time t are themselves conditionally independent Brownian disks given this boundary length process.

The results of this paper will be used in another work of the authors to show that the scaling limit of percolation on random quadrangulations is given by SLE_6 on $\sqrt{8/3}$ -LQG with respect to the Gromov-Hausdorff-Prokhorov-uniform topology, the natural analog of the Gromov-Hausdorff topology for curve-decorated metric measure spaces.

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1 Introduction

1.1 Overview

The Schramm-Loewner evolution (SLE) was introduced by Schramm [Sch00] to describe the scaling limits of the interfaces which arise in discrete two-dimensional models, such as loop-erased random walk, critical percolation interfaces, and the peano curve of the uniform spanning tree. The form of SLE was derived by Schramm using what is now called its *conformal Markov property*. This says that if η is an SLE $_{\kappa}$ curve in \mathbb{H} from 0 to ∞ then for each time t the conditional law of $\eta|_{[t,\infty)}$ given $\eta|_{[0,t]}$ is given by the conformal image of an SLE $_{\kappa}$ in \mathbb{H} from 0 to ∞ . To show that a curve is an SLE one need only show that this property is satisfied. Beyond the initial derivation of SLE, this perspective has been very powerful for the purpose of establishing properties of SLE. Moreover, results of this type have been shown to characterize other SLE-related processes. For example, it was shown by Sheffield and Werner [SW12] that the so-called simple *conformal loop ensembles* (CLE), the loop form of SLE $_{\kappa}$ for $\kappa \in (8/3, 4]$, are similarly characterized by a variant of the conformal Markov property.

The purpose of the present work is to establish a characterization of SLE in the spirit of the conformal Markov property but in the context of *Liouville quantum gravity* (LQG). It will be given in terms of a set of conditions which one would naturally expect any subsequential limit of certain statistical physics models on a random planar map to satisfy. This characterization will in turn play an important role in work of the authors to show that the scaling limit of (critical) percolation on random quadrangulations is given by SLE₆ on $\sqrt{8/3}$ -LQG. This is carried out in [GM17b] (building also on [GM17c]) in which we show that the subsequential limits of percolation on random quadrangulations exist and satisfy the hypotheses of our characterization theorem.

Formally, an LQG surface for $\gamma \in (0, 2)$ is a random Riemann surface parameterized by a domain $D \subset \mathbb{C}$ whose Riemannian metric tensor is $e^{\gamma h(z)} dx \otimes dy$, where h is some variant of the Gaussian free field (GFF) on D and $dx \otimes dy$ is the Euclidean metric tensor. This does not make rigorous sense since h is a distribution, not a function, so does not take values at points. However, Duplantier and Sheffield [DS11] showed that one can make rigorous sense of the volume form $\mu_h = "e^{\gamma h(z)} dz"$ associated with an LQG surface as a random measure on D via a regularization procedure. One can similarly define a random length measure ν_h associated with an LQG surface, which is defined on certain curves in D including ∂D and independent Schramm-Loewner evolution [Sch00] (SLE $_{\kappa}$)-type curves for $\kappa = \gamma^2$ [She16a].

The measures μ_h and ν_h satisfy a conformal covariance formula: if $f : D \rightarrow \tilde{D}$ is a conformal map and

$$\tilde{h} = h \circ f^{-1} + Q \log |(f^{-1})'|, \quad \text{for } Q = \frac{2}{\gamma} + \frac{\gamma}{2} \quad (1.1)$$

then f pushes forward μ_h to $\mu_{\tilde{h}}$ and ν_h to $\nu_{\tilde{h}}$ (in fact, this holds a.s. for all choices of conformal map f simultaneously [SW16]). Hence one can rigorously define an LQG surface as an equivalence class of pairs (D, h) consisting of a domain $D \subset \mathbb{C}$ and a distribution h on D , with two such pairs declared to be equivalent if the distributions are related by a conformal map as in (1.1) which extends to a homeomorphism $D \cup \partial D \rightarrow \tilde{D} \cup \partial \tilde{D}$. In other words, a quantum surface is an equivalence class of measure spaces modulo conformal maps. This

definition does not require that D be simply connected or even connected. In particular, one can make sense of quantum surfaces consisting of a string of beads, each of which is itself a quantum surface homeomorphic to the unit disk.

It is expected that one can also make sense of an LQG surface as a random *metric space*. So far this has only been accomplished in the special case when $\gamma = \sqrt{8/3}$ in the series of works [MS15c, MS15a, MS15b, MS16b, MS16c].

LQG surfaces arise as scaling limits of various random planar map models. LQG for $\gamma = \sqrt{8/3}$ corresponds to the scaling limit of uniform random planar maps, and other values of γ arise by sampling a planar map with probability proportional to the partition function of an appropriate γ -dependent statistical mechanics model on the map. For example, weighting by the number of spanning trees corresponds to $\gamma = \sqrt{2}$, weighting by the partition function of the Ising model corresponds to $\gamma = \sqrt{3}$, and weighting by the number of bipolar orientations corresponds to $\gamma = \sqrt{4/3}$. So far, scaling limit results for random planar maps toward LQG have been obtained in the Gromov-Hausdorff topology for $\gamma = \sqrt{8/3}$ [Le 13, Mie13] (together with [MS15c, MS15a, MS15b, MS16b, MS16c]) and in the so-called peanosphere sense, which relies on the main theorem of [DMS14], for all values of $\gamma \in (0, 2)$ [She16b, KMSW15, GKMW16].

It is natural to consider a γ -LQG surface decorated by an independent SLE_{κ} -type curve for $\kappa = \gamma^2 \in (0, 4)$ or $\kappa' = 16/\gamma^2 > 4$.¹ One reason why this is natural is that such curve-decorated quantum surfaces describe the scaling limits of statistical mechanics models on random planar maps in the γ -LQG universality class. Indeed, scaling limits in the peanosphere sense are really statements about the convergence of random planar maps decorated by a space-filling curve toward LQG decorated by space-filling SLE (as defined in [MS13]). See also [GM16a] for a scaling limit result for random quadrangulations decorated by a self-avoiding walk toward $\text{SLE}_{8/3}$ -decorated $\sqrt{8/3}$ -LQG in a variant of the Gromov-Hausdorff topology.

In the continuum, there are a number of theorems which describe γ -LQG surfaces decorated by independent SLE_{κ} - or $\text{SLE}_{\kappa'}$ -type curves [DMS14, MS15c]. One such theorem which will be important in this paper is the *peanosphere* or *mating of trees* construction of [DMS14], which we now briefly describe. Suppose $(\mathbb{C}, h, 0, \infty)$ is a γ -quantum cone, a particular type of γ -LQG surface with two marked points which describes the local behavior of any γ -LQG surface near a point sampled from the γ -LQG area measure. Let η' be a whole-plane space-filling $\text{SLE}_{\kappa'}$ curve from ∞ to ∞ for $\kappa' = 16/\gamma^2$, independent from h . In the case when $\kappa' \geq 8$, η' is just a two-sided variant of chordal $\text{SLE}_{\kappa'}$. In the case when $\kappa' \in (4, 8)$ (so ordinary $\text{SLE}_{\kappa'}$ is not space-filling), η' is obtained from a two-sided variant of ordinary whole-plane $\text{SLE}_{\kappa'}$ by iteratively filling in the bubbles which it surrounds by $\text{SLE}_{\kappa'}$ -type curves.

Suppose we parameterize η' by γ -quantum mass with respect to h , so that $\mu_h(\eta'([s, t])) = t - s$ for each $s < t$ and $\eta'(0) = 0$. For $t \geq 0$, let L_t (resp. R_t) be the net change in the ν_h -length of the left (resp. right) outer boundary of $\eta'((-\infty, t])$ relative to time 0. Then by [DMS14, Theorem 1.13], there is a universal constant $\alpha = \alpha(\gamma) > 0$ such that $Z_t := (L_t, R_t)$ is a pair of correlated Brownian motions with

$$\text{Var } L_t = \text{Var } R_t = \alpha|t| \quad \text{and} \quad \text{Cov}(L_t, R_t) = -\alpha \cos\left(\frac{\pi\gamma^2}{4}\right), \quad \forall t \in \mathbb{R}. \quad (1.2)$$

In other words, η' is an embedding into \mathbb{C} of the space-filling curve on an infinite-volume peanosphere, a random curve-decorated topological measure space obtained by gluing together a pair of correlated Brownian motions (see Figure 2). It is shown in [DMS14, Theorem 1.14] that Z a.s. determines (h, η') , modulo scaling and rotation.

As mentioned above, the goal of the present paper is to show that $\text{SLE}_{\kappa'}$ for $\kappa' \in (4, 8)$ on γ -LQG for $\gamma \in (\sqrt{2}, 2)$ is characterized by certain natural properties, in the spirit of Schramm [Sch00]. Our main result, Theorem 1.1 (c.f. Theorem 2.4 for a stronger statement), roughly states the following. Suppose that $((\mathbb{C}, \tilde{h}, 0, \infty), \tilde{\eta}')$ is a γ -quantum cone decorated by a space-filling curve which differs from the pair (h, η') above by a quantum area- and quantum length-preserving homeomorphism and which satisfies a quantum analog of the conformal Markov property. Then $(\tilde{h}, \tilde{\eta}') = (h, \eta')$, modulo rotation and scaling. We will also establish an analogous characterization in the setting of chordal SLE in Theorem 6.2.

Our characterization theorems are closely related in spirit to the question of whether $\text{SLE}_{\kappa'}$ for $\kappa' \in (4, 8)$ is *conformally removable*, i.e., every homeomorphism which is conformal off the range of such a path is conformal

¹Here and throughout this paper we use the imaginary geometry [MS16d, MS16e, MS16a, MS13] convention of writing κ for the SLE parameter when $\kappa \in (0, 4)$ and $\kappa' = 16/\kappa$ for the dual parameter.

everywhere. It is in general a difficult question to determine whether a fractal carpet like the range of $\text{SLE}_{\kappa'}$ is conformally removable. See Section 2.2.3 for some additional discussion of this point. Our results allow us to circumvent dealing with the removability question directly when trying to show that certain random curves are $\text{SLE}_{\kappa'}$'s.

Recall that a $\sqrt{8/3}$ -LQG surface admits a metric [MS15c, MS15a, MS15b, MS16b, MS16c]. In the special case when $\gamma = \sqrt{8/3}$, we can re-phrase our characterization theorems in terms of this metric space structure, and thereby in terms of so-called Brownian surfaces, which we discuss just below. The metric space version of our chordal SLE characterization theorem allow us to identify the scaling limit of percolation on random quadrangulations in [GM17b].

The *Brownian map* is a random metric measure space, constructed via a continuum analog of the Schaeffer bijection [Sch97], which arises as the scaling limit of uniform random planar maps on the sphere [Le 13, Mie13]. A *Brownian surface* is a random metric measure space which locally looks like the Brownian map. Such surfaces include the Brownian plane [CL14], the Brownian disk [BM15], and the Brownian half-plane [GM16c, BMR16]. Certain $\sqrt{8/3}$ -LQG surfaces are equivalent as metric measure spaces to these Brownian surfaces.

- The quantum sphere is equivalent to the Brownian map.
- The $\sqrt{8/3}$ -quantum cone (which we recall is the surface arising in the peanosphere construction) is equivalent to the Brownian plane.
- The quantum disk is equivalent to the Brownian disk.
- The $\sqrt{8/3}$ -quantum wedge is equivalent to the Brownian half-plane.

It is shown in [MS16c] that the metric measure space structure of a $\sqrt{8/3}$ -LQG surface a.s. determines its quantum surface structure. Hence the results of [MS15c, MS15a, MS15b, MS16b, MS16c] can be viewed as endowing a Brownian surface with a canonical conformal structure, and constructing numerous additional Brownian surfaces.

In particular, these results allow us to make sense of $\text{SLE}_{8/3}$ - or SLE_6 -type curves on Brownian surfaces (by embedding the surface into a domain in \mathbb{C} , then drawing an independent SLE curve). It is natural to expect that such curves respectively arise as the scaling limit of self-avoiding walks and percolation explorations on uniform random planar maps. In the former case, this was proven in [GM16a], building on [GM16b, GM16c] (see also [GM17c] for the finite-volume case), and in the latter case this is proven in [GM17b], building on [GM17c, GM17a] and the present work. That is, the results of [GM16a, GM17b] allow us to say that the definitions of $\text{SLE}_{8/3}$ and SLE_6 on Brownian surfaces which come from $\sqrt{8/3}$ -LQG are the correct ones because they describe the scaling limits of the corresponding discrete models.

Since the conformal structure of a Brownian surface does not depend on the metric measure space structure in an explicit way, the $\sqrt{8/3}$ -LQG metric construction does not yield an explicit description of an SLE on a Brownian surface which depends only on the metric measure space structure. The results of the present paper allow us to describe a whole-plane space-filling SLE_6 on the Brownian plane (Theorem 7.1) or a chordal SLE_6 on the Brownian disk (Theorem 7.2) by means of a list of conditions which depend only on the metric measure space structure.

The conditions in the above description are possible to verify for subsequential limits of percolation models on random planar maps. Hence our results are a key tool for showing that such percolation models converge to SLE_6 -decorated $\sqrt{8/3}$ -LQG surfaces in the Gromov-Hausdorff-Prokhorov-uniform topology [GM16c], the natural analog of the Gromov-Hausdorff topology for curve-decorated metric measure spaces. In [GM17b], we will use Theorem 7.2 of the present paper to identify the scaling limit of percolation explorations on uniform random quadrangulations with simple boundary as chordal SLE_6 on the Brownian disk or Brownian half-plane (depending on whether the quadrangulation has finite or infinite boundary). We expect that it should be possible to prove similar results for other percolation models on random planar maps using the same strategy.

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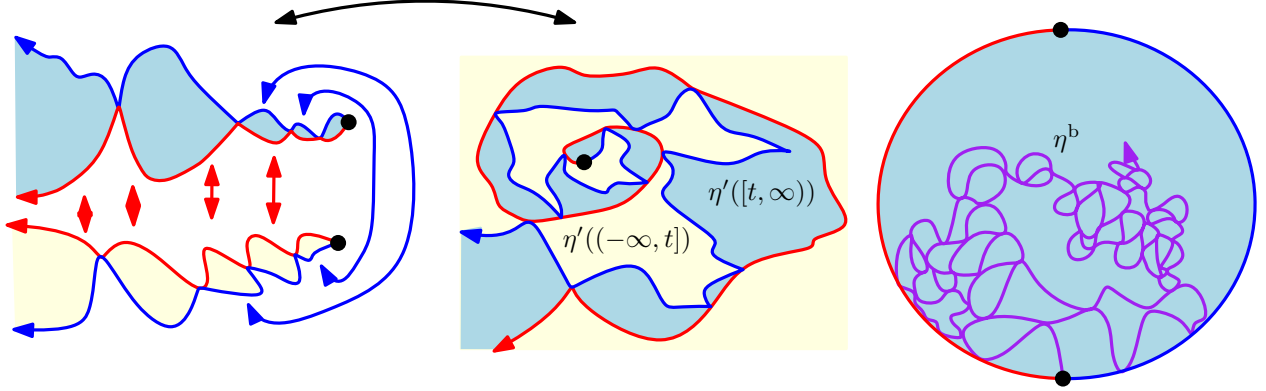


Figure 1: **Left and middle:** A whole-plane space-filling $\text{SLE}_{\kappa'}$ for $\kappa' \in (4, 8)$ on an independent γ -quantum cone, which is the object characterized in Theorem 1.1. The past $\eta'((-\infty, t])$ is shown in yellow and the future $\eta'([t, \infty))$ is shown in blue. Restricting the field h to each of these sets gives two independent beaded quantum surfaces, each of which has the law of a $\frac{3\gamma}{2}$ -quantum wedge, which can be conformally welded together according to quantum length along their boundaries to obtain the γ -quantum cone $(\mathbb{C}, h, 0, \infty)$. In the special case when $\gamma = \sqrt{8/3}$, the $\sqrt{8/3}$ -quantum cone admits a metric measure space structure under which it is equivalent to the Brownian plane. In this case, Theorem 1.1 can be re-phrased as a characterization theorem for whole-plane space-filling SLE_6 on the Brownian plane; see Theorem 7.1 for a stronger variant of this characterization. **Right:** A chordal $\text{SLE}_{\kappa'}$ on an independent bead of a $\frac{3\gamma}{2}$ -quantum wedge (i.e., a quantum surface whose law is the same as the conditional law of one of the connected components of the quantum surface parameterized by $\eta'([t, \infty))$ given its quantum area and left/right quantum boundary lengths). This is the setting of our other quantum surface characterization theorem, Theorem 6.2. When $\gamma = \sqrt{8/3}$, a single bead of a $\frac{3\gamma}{2}$ -quantum wedge is the same as the quantum disk, which in turn is the same as the Brownian disk when equipped with its $\sqrt{8/3}$ -LQG metric and area measure. This gives rise to a characterization of chordal SLE_6 on the Brownian disk; see Theorem 7.2.

1.2 Main results

The first main result of this paper, from which our other results will follow, is a characterization of whole-plane space-filling $\text{SLE}_{\kappa'}$ on an independent γ -quantum cone; this is the setting of the peanosphere construction of [DMS14]. Here we will state a simplified version of our characterization theorem; the full version (which has weaker but more complicated hypotheses) appears as Theorem 2.4 below. See Figure 1 for an illustration.

For the statement, we recall that a $\frac{3\gamma}{2}$ -quantum wedge (or weight- $(2-\gamma^2/2)$ quantum wedge) for $\gamma \in (\sqrt{2}, 2)$ is a quantum surface consisting of an infinite Poissonian string of beads, each of which is a finite-volume doubly marked quantum surface homeomorphic to the unit disk \mathbb{D} (see [DMS14, Definition 4.13] or Section 2.2.2). In the special case when $\gamma = \sqrt{8/3}$, these beads are in fact doubly marked quantum disks. The $\frac{3\gamma}{2}$ -quantum wedge arises as the quantum surface parameterized by $\eta'([t, \infty))$ when η' is an independent space-filling $\text{SLE}_{\kappa'}$ on an independent γ -quantum cone.

Theorem 1.1 (Whole-plane space-filling SLE characterization, simplified version). *Let $\kappa' \in (4, 8)$ and $\gamma = 4/\sqrt{\kappa'} \in (\sqrt{2}, 2)$. Suppose that $(\tilde{h}, \tilde{\eta}', Z)$ is a given coupling where \tilde{h} is an embedding into $(\mathbb{C}, 0, \infty)$ of a γ -quantum cone, $\tilde{\eta}' : \mathbb{R} \rightarrow \mathbb{C}$ is a random continuous curve parameterized by γ -quantum mass with respect to \tilde{h} with $\tilde{\eta}'(0) = 0$, and $Z = (L, R)$ is a correlated two-dimensional Brownian motion with variances and covariance as in (1.2). Assume that the following conditions are satisfied.*

1. (Markov property) *For each $t \in \mathbb{R}$, the doubly marked quantum surface $(\tilde{\eta}'([t, \infty)), \tilde{h}|_{\tilde{\eta}'([t, \infty))}, \tilde{\eta}'(t), \infty)$ is a $\frac{3\gamma}{2}$ -quantum wedge and the curve-decorated quantum surfaces $(\tilde{\eta}'((-\infty, t]), \tilde{h}|_{\tilde{\eta}'((-\infty, t])}, \tilde{\eta}'|_{(-\infty, t]})$ and $(\tilde{\eta}'([t, \infty)), \tilde{h}|_{\tilde{\eta}'([t, \infty))}, \tilde{\eta}'|_{[t, \infty)})$ are independent.*
2. (Topology and consistency) *The curve-decorated topological space $(\mathbb{C}, \tilde{\eta}')$ is equivalent to the infinite-volume peanosphere generated by Z . Equivalently, if $((\mathbb{C}, h, 0, \infty), \eta')$ is the pair consisting of a γ -quantum cone and an independent space-filling $\text{SLE}_{\kappa'}$ from ∞ to ∞ parameterized by γ -quantum mass with respect*

to h which is determined by Z via [DMS14, Theorem 1.14], then there is a homeomorphism $\Phi : \mathbb{C} \rightarrow \mathbb{C}$ with $\Phi \circ \eta' = \tilde{\eta}'$. Moreover, Φ a.s. pushes forward the γ -quantum length measure on $\partial\eta'([t, \infty))$ with respect to h to the γ -quantum length measure on $\partial\tilde{\eta}'([t, \infty))$ with respect to \tilde{h} for each $t \in \mathbb{Q}$.

Then $(\tilde{h}, \tilde{\eta}')$ is an embedding into $(\mathbb{C}, 0, \infty)$ of a γ -quantum cone together with an independent whole-plane space-filling $\text{SLE}_{\kappa'}$ from ∞ to ∞ parameterized by γ -quantum mass with respect to \tilde{h} . In fact, the map Φ of condition 2 is a.s. given by multiplication by a complex number.

Condition 1 is a quantum version of the conformal Markov property for the pair $(\tilde{h}, \tilde{\eta}')$, which is analogous to the conformal Markov property of ordinary $\text{SLE}_{\kappa'}$. Unlike in the case of ordinary $\text{SLE}_{\kappa'}$, however, we do *not* assume that the law of the curve-decorated quantum surfaces $(\tilde{\eta}'([t, \infty)), \tilde{h}|_{\tilde{\eta}'([t, \infty))}, \tilde{\eta}'|_{[t, \infty)})$ is stationary in t . We also emphasize that we only assume an independence statement for curve-decorated quantum surfaces, i.e. equivalence classes of triples consisting of a domain, field, and curve modulo conformal maps. In particular, $(\tilde{\eta}'([t, \infty)), \tilde{h}|_{\tilde{\eta}'([t, \infty))}, \tilde{\eta}'|_{[t, \infty)})$ does not determine the particular embedding of the curve $\tilde{\eta}'|_{[t, \infty)}$ into \mathbb{C} .

The first part of condition 2 enables us to define the γ -quantum length measure with respect to \tilde{h} on $\partial\tilde{\eta}'([t, \infty))$ simultaneously for all $t \in \mathbb{R}$ in such a way that the last part of condition 2 holds simultaneously a.s. for all $t \in \mathbb{R}$. This is because any arc of $\partial\tilde{\eta}'([t, \infty))$ which is at positive distance from $\tilde{\eta}'(t)$ is contained in $\partial\tilde{\eta}'([s, \infty))$ for some rational $s \geq t$, so we can define the length measure on this arc in terms of the length measure on $\partial\tilde{\eta}'([s, \infty))$. This definition does not depend on the particular choice of rational $s \geq t$ by the last part of condition 2. Hence Z is the left/right boundary length process for the pair $(\tilde{\eta}', \tilde{h})$.

It is clear from the results of [DMS14] (see in particular the proof of [DMS14, Lemma 9.2] for condition 1) that the conditions of Theorem 2.4 are satisfied in the case when $(\tilde{h}, \tilde{\eta}') = (h, \eta')$ is an embedding into $(\mathbb{C}, 0, \infty)$ of a γ -quantum cone together with an independent whole-plane space-filling $\text{SLE}_{\kappa'}$ from ∞ to ∞ , parameterized by γ -quantum mass. Hence Theorem 2.4 provides a list of conditions which uniquely characterize the law of the pair (h, η') .

We will actually prove a stronger version of Theorem 1.1, with weaker but more complicated assumptions; see Theorem 2.4 below. This latter version of the theorem is the one which we expect to be more useful in practice for identifying the laws of subsequential scaling limits of discrete models. The stronger characterization result of Theorem 2.4 will be used to prove several other characterization theorems, which are stated in Sections 6 and 7. In particular, in Section 6 we will prove Theorem 6.2, a variant of Theorem 1.1 for chordal (non-space-filling) $\text{SLE}_{\kappa'}$ on a single bead of a $\frac{3\gamma}{2}$ -quantum wedge.

In the special case when $\kappa' = 6$ ($\gamma = \sqrt{8/3}$), a $\sqrt{8/3}$ -LQG surface has a metric measure space structure which a.s. determines the same information as its quantum surface structure [MS15b, MS16b, MS16c] (see Section 2.2.4 below for more details). In Section 7, we will use this equivalence to re-phrase our characterization results Theorems 2.4 and 6.2 in the case when $\kappa' = 6$ in terms of metric measure spaces rather than quantum surfaces. This will lead to Theorem 7.1, a characterization of whole-plane space-filling SLE_6 on the Brownian plane and Theorems 7.2 and Theorem 7.12, two characterizations of chordal SLE_6 on the Brownian disk with different choices of parameterization.

These metric characterization theorems allow us to identify the laws of curve-decorated metric measure spaces, e.g., those arising as subsequential limits of curve-decorated random planar maps in the Gromov-Hausdorff-Prokhorov-uniform (GHPU) topology [GM16c], the natural variant of the Gromov-Hausdorff topology for curve-decorated metric measure spaces. Theorem 7.12 in particular will be used in [GM17b] to show that the exploration path for percolation on a random quadrangulation with simple boundary converges in the scaling limit in the GHPU topology to SLE_6 on a Brownian disk.

1.3 Outline

In this subsection we provide a moderately detailed overview of the content of the remainder of the paper and the proofs of our main results.

We start in Section 2 by reviewing some background on LQG surfaces, space-filling $\text{SLE}_{\kappa'}$, and the relationships between them. We also state a stronger version of Theorem 1.1, namely Theorem 2.4.

The next three sections are devoted to the proof of Theorem 2.4. The basic idea of the proof has some similarities with that of [DMS14, Theorem 1.14], in that we will swap out quantum surfaces parameterized by small segments of certain space-filling curves one at a time and bound the distortion of the conformal maps between certain embeddings of these surfaces into \mathbb{C} .

In Section 3, we will first define several objects in terms of the γ -quantum cone/space-filling $\text{SLE}_{\kappa'}$ pair (h, η') from Theorem 2.4 as well as their counterparts with the given pair $(\tilde{h}, \tilde{\eta}')$ in place of (h, η') . These objects include the beaded quantum surfaces $\mathcal{S}_{a,b}$ (resp. $\tilde{\mathcal{S}}_{a,b}$) obtained by restricting h (resp. \tilde{h}) to $\eta'([a, b])$ (resp. $\tilde{\eta}'([a, b])$) as well as non-space-filling curves $\eta_{a,b}$ (resp. $\tilde{\eta}_{a,b}$) obtained from $\eta'|_{[a,b]}$ (resp. $\tilde{\eta}'|_{[a,b]}$) by skipping the intervals of time during which it is filling in a bubble. We will then prove several statements about the laws of these objects which build on the hypotheses of Theorem 2.4. In particular, we will establish the following.

- The quantum surface $\mathcal{S}_{a,b}^0$ parameterized by the bubbles cut out by $\eta_{a,b}$ and the quantum surface $\tilde{\mathcal{S}}_{a,b}^0$ parameterized by the bubbles cut out by $\tilde{\eta}_{a,b}$ have the same law (both of these surfaces are collections of independent quantum disks if we condition on the areas and boundary lengths of their interior connected components; c.f. Propositions 3.9 and 3.12).
- More generally, for $t \in [a, b]$ the joint law of the quantum surface parameterized by the bubbles cut out by $\eta_{a,b}|_{[0,t]}$ and the quantum surface parameterized by the region traced by η' after it finishes filling in all of these bubbles is the same as the joint law of the analogous pair of quantum surfaces defined in terms of $(\tilde{h}, \tilde{\eta}')$ (Proposition 3.17).

In Section 4, we will prove for given $n \in \mathbb{N}$ an estimate for the sum of the squared diameters of the time length- $1/n$ segments of the curve $\eta'|_{[a,b]}$ which intersect the $\text{SLE}_{\kappa'}$ -type curve $\eta_{a,b}$ of Section 3 (Proposition 4.1). In particular, we will show that the expectation of this sum tends to zero faster than some negative power of n . (This is the step in the argument that does not extend to $\kappa' \geq 8$ because in this case the sum of the squared diameters of the time length- $1/n$ curve segments will typically be of constant order.) Combined with the distortion estimates described in [DMS14, Section 10] (see in particular [DMS14, Lemma 10.5]) this estimate will eventually allow us to bound how much the conformal maps in the curve-swapping argument mentioned above deviate from the identity.

The proof of Proposition 4.1 proceeds by way of two other results which are of independent interest: a KPZ-type bound for the Lebesgue measure of the ϵ -neighborhood of a subset of \mathbb{C} which is independent from h in terms of the expected number of ϵ -length intervals needed to cover its pre-image under η' (which is a variant of the Hausdorff dimension relation [GHM15, Theorem 1.1]); and an estimate for the expected number of ϵ -length intervals needed to cover the set of times $t \in [a, b]$ which are not contained in any $\pi/2$ -cone interval for Z in $[a, b]$, which is the pre-image of $\eta_{a,b}$ under η' (Proposition 4.7). We will also introduce in Section 4.4 a regularity condition which is needed in order to apply Proposition 4.1 in the next section.

Section 5 contains the curve-swapping argument which will eventually lead to a proof of Theorem 2.4. The main step is to show that for fixed $a, b \in \mathbb{R}$ with $a < b$, the quantum surface $\mathcal{S}_{a,b}$ parameterized by the space-filling $\text{SLE}_{\kappa'}$ segment $\eta'([a, b])$ has the same law as the surface $\tilde{\mathcal{S}}_{a,b}$ parameterized by the corresponding segment $\tilde{\eta}'([a, b])$ of our candidate curve (Proposition 5.1). To prove this, set $t_{n,k} := a + \frac{k}{n}(b-a)$ for $k \in [0, n]_{\mathbb{Z}}$. We will define for each $n \in \mathbb{N}$ and each $k \in [0, n]_{\mathbb{Z}}$ a quantum surface $\check{\mathcal{C}}_{n,k}$, which has the law of a γ -quantum cone, decorated by a non-space-filling curve $\check{\eta}_{n,k}$ (an analog of $\eta_{a,b}$) and a space-filling curve $\check{\eta}'_{n,k}$ (an analog of η') with the following properties.

- The sub-surface of $\check{\mathcal{C}}_{n,0}$ (resp. $\check{\mathcal{C}}_{n,n}$) parameterized by the curve segment $\check{\eta}'_{n,0}([a, b])$ (resp. $\check{\eta}'_{n,n}([a, b])$) has the same law as $\tilde{\mathcal{S}}_{a,b}$ (resp. $\mathcal{S}_{a,b}$).
- The triples $(\check{\mathcal{C}}_{n,k}, \check{\eta}_{n,k}, \check{\eta}'_{n,k})$ are all topologically equivalent. In fact, there exists for each $k \in [1, n]_{\mathbb{Z}}$ a homeomorphism $f_{n,k} : \check{\mathcal{C}}_{n,k} \rightarrow \check{\mathcal{C}}_{n,k-1}$ which satisfies $f_{n,k} \circ \check{\eta}_{n,k} = \check{\eta}_{n,k-1}$ and $f_{n,k} \circ \check{\eta}'_{n,k} = \check{\eta}'_{n,k-1}$ and which is conformal on $\mathbb{C} \setminus \check{\eta}_{n,k}([t_{n,k-1}, t_{n,k}])$ where $t_{n,k} = a + \frac{k}{n}(b-a)$ (Lemma 5.3).
- The map $f_{n,k}$ is the identity map if $\check{\eta}'_{n,k}([t_{n,k-1}, t_{n,k}])$ does not intersect $\check{\eta}_{n,k}$.

The triples $(\check{\mathcal{C}}_{n,k}, \check{\eta}_{n,k}, \check{\eta}'_{n,k})$ will be constructed from $(\tilde{h}, \tilde{\eta}')$ by (roughly speaking) the following procedure. First impose a conformal structure on the closure of the union of the bubbles cut out by the curve $\tilde{\eta}_{a,b}$ run up to time $t_{n,k}$ in the same manner that one imposes a conformal structure on the bubbles cut out by $\eta_{a,b}$ run up to time $t_{n,k}$ to get a sub-surface of $\mathcal{S}_{a,b}$. This gives us a quantum surface, which we then conformally weld to the quantum surface parameterized by the past of η' and the future of $\tilde{\eta}'$ to get $\check{\mathcal{C}}_{n,k}$. The curve $\check{\eta}_{n,k}$ is

the concatenation of the gluing interface of the bubbles and the future curve $\tilde{\eta}_{a,b}|_{[t_{n,k},b]}$. The curve $\tilde{\eta}'_{n,k}$ is obtained by filling in the bubbles cut out by $\tilde{\eta}_{n,k}$ with conformal images of segments of $\tilde{\eta}'$. The results of Section 3 will be used to ensure that this construction makes sense and that the above conditions are satisfied.

We will then use the results of Section 4 together with the distortion estimate [DMS14, Lemma 10.5] to show that the sum of the deviations of the maps $f_{n,k}$ from the identity tends to 0 as $n \rightarrow \infty$. Composing these maps will then give us the desired equality in law $\mathcal{S}_{a,b} \stackrel{d}{=} \tilde{\mathcal{S}}_{a,b}$. Theorem 2.4 is obtained by applying this equality in law to the quantum surfaces $\{\mathcal{S}_{(j-1)\epsilon, j\epsilon}\}_{j \in \mathbb{N}}$ and $\{\tilde{\mathcal{S}}_{(j-1)\epsilon, j\epsilon}\}_{j \in \mathbb{N}}$ to get that the joint laws of the collections of points $\{\eta'(j\epsilon)\}_{j \in \mathbb{N}}$ and $\{\tilde{\eta}'(j\epsilon)\}_{j \in \mathbb{N}}$ are the same, then sending $\epsilon \rightarrow 0$.

The last two sections of the paper deduce additional characterization theorems which are consequences of Theorem 2.4. In Section 6, we will use Theorem 2.4 to prove Theorem 6.2, a characterization theorem for chordal $\text{SLE}_{\kappa'}$ on a single bead of $\frac{3\gamma}{2}$ -quantum wedge.

In Section 7, we will restrict attention to the special case when $\kappa' = 6$ and prove metric space versions of Theorems 2.4 and 6.2 (Theorem 7.1, and Theorems 7.2 and 7.12 respectively) using the fact that the metric measure space structure and the quantum surface structure of a $\sqrt{8/3}$ -LQG surface a.s. determine each other [MS16c].

2 Preliminaries

In this section we will review the definitions of the various objects involved in the statements and proofs of our main results, including LQG surfaces, whole-plane space-filling $\text{SLE}_{\kappa'}$, and the peanosphere construction, and provide references to where more details can be found. We also state in Section 2.4 a stronger version of Theorem 1.1, which is the version which will be proven in this paper and used to deduce our other results.

Throughout this paper we fix parameters

$$\gamma \in (0, 2), \quad \kappa = \gamma^2 \in (0, 4), \quad \text{and} \quad \kappa' = \frac{16}{\gamma^2} \in (4, 8). \quad (2.1)$$

We will often restrict attention to the case when $\gamma \in (\sqrt{2}, 2)$ (equivalently $\kappa \in (2, 4)$ and $\kappa' \in (4, 8)$) and we will occasionally further restrict to the special case where $\gamma = \sqrt{8/3}$ (equivalently $\kappa = 8/3$ and $\kappa' = 6$), since this is the only case where a metric on γ -LQG has been constructed.

2.1 Basic notation

Here we record some basic notation which we will use throughout this paper.

We write \mathbb{N} for the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

For $a < b \in \mathbb{R}$, we define the discrete intervals $[a, b]_{\mathbb{Z}} := [a, b] \cap \mathbb{Z}$ and $(a, b)_{\mathbb{Z}} := (a, b) \cap \mathbb{Z}$.

If a and b are two quantities, we write $a \preceq b$ (resp. $a \succeq b$) if there is a constant $C > 0$ (independent of the parameters of interest) such that $a \leq Cb$ (resp. $a \geq Cb$). We write $a \asymp b$ if $a \preceq b$ and $a \succeq b$.

If a and b are two quantities which depend on a parameter x , we write $a = o_x(b)$ (resp. $a = O_x(b)$) if $a/b \rightarrow 0$ (resp. a/b remains bounded) as $x \rightarrow 0$, or as $x \rightarrow \infty$, depending on context. We write $a = o_x^\infty(b)$ if $a = o_x(b^s)$ for every $s \in \mathbb{R}$.

Unless otherwise stated, all implicit constants in \asymp , \preceq , and \succeq and $O_x(\cdot)$ and $o_x(\cdot)$ errors involved in the proof of a result are required to depend only on the auxiliary parameters that the implicit constants in the statement of the result are allowed to depend on.

2.2 Liouville quantum gravity surfaces

We will now review the definition and basic properties of Liouville quantum gravity surfaces. Throughout, we fix γ, κ , and κ' as in (2.1).

2.2.1 LQG coordinate change and LQG measures

For $k \in \mathbb{N}_0$, a γ -Liouville quantum gravity (LQG) surface with k marked points is an equivalence class \mathcal{S} of $(k+2)$ -tuples (D, h, x_1, \dots, x_k) , where $D \subset \mathbb{C}$ is an open set (not necessarily connected or simply connected), h is a distribution on D , and $x_1, \dots, x_k \in D \cup \partial D$. Two such $(k+2)$ -tuples (D, h, x_1, \dots, x_k) and $(\tilde{D}, \tilde{h}, \tilde{x}_1, \dots, \tilde{x}_k)$ are equivalent (i.e., they correspond to the same quantum surface) if there is a conformal map $f : D \rightarrow \tilde{D}$ such that

$$\tilde{h} = h \circ f^{-1} + Q \log |(f^{-1})'| \quad \text{and} \quad f(x_j) = \tilde{x}_j, \quad \forall j \in [1, k]_{\mathbb{Z}}, \quad \text{where } Q = \frac{2}{\gamma} + \frac{\gamma}{2}; \quad (2.2)$$

and f extends to a homeomorphism from $D \cup \partial D$ to $\tilde{D} \cup \partial \tilde{D}$.

If (D, h, x_1, \dots, x_k) is a particular choice of equivalence class representative, we say that h is an *embedding* of the quantum surface \mathcal{S} into (D, h, x_1, \dots, x_k) . We will sometimes slightly abuse notation by writing $\mathcal{S} = (D, h, x_1, \dots, x_k)$ when it is clear that we are referring to the whole quantum surface rather than a particular equivalence class representative.

A *sub-surface* of a quantum surface $\mathcal{S} = (D, h)$ is a quantum surface $\mathcal{S}' = (D', h|_{D'})$ for a domain $D' \subset D$.

For all of the quantum surfaces we will consider in this paper, the distribution h will be some variant of the Gaussian free field (GFF) on D . See [She07, SS13, She16a, MS16d, MS13] for more on the GFF. It is shown in [DS11] that such a distribution h induces a Borel measure μ_h on D , called the γ -quantum area measure; and a measure ν_h called the γ -quantum length measure which is defined on ∂D as well as on certain curves in D , including SLE $_{\kappa}$ -type curves which are independent from h for $\kappa = \gamma^2$ [She16a].

It is shown in [DS11] that if f is a fixed *deterministic* conformal map which satisfies (2.2), then a.s. $f_*\mu_h = \mu_{\tilde{h}}$ and $f_*\nu_h = \nu_{\tilde{h}}$. In [SW16], it is shown that the same is true a.s. for all conformal maps f simultaneously. In particular, the measures μ_h and ν_h are well-defined functionals of the quantum surface. Furthermore, it is shown in [BSS14] that the γ -quantum area measure μ_h a.s. determines h . Hence a quantum surface can equivalently be viewed as an equivalence class of measure spaces, modulo conformal maps.

2.2.2 Quantum disks, cones, and wedges

In this paper we will have occasion to consider several particular types of quantum surfaces which are defined in [DMS14]. Here we give a brief review of these particular types of quantum surfaces, with references to where more information can be found. Throughout this subsection we fix $\gamma \in (0, 2)$ and let Q be as in (2.2).

A *quantum disk* is a finite-volume quantum surface with boundary (\mathbb{D}, h) defined in [DMS14, Definition 4.21], which can be taken to have fixed area, boundary length, or both. A *singly (resp. doubly) marked quantum disk* is a quantum disk together with one (resp. two) marked points in $\partial \mathbb{D}$ sampled uniformly (and independently) from the γ -LQG boundary length measure ν_h . Note that the marked points in [DMS14, Definition 4.21] correspond to the points $\pm\infty$ in the infinite strip, and are shown to be sampled uniformly and independently from μ_h in [DMS14, Proposition 5.7].

For $\alpha < Q$, an α -quantum cone is a doubly-marked quantum surface $(\mathbb{C}, h, 0, \infty)$ defined precisely in [DMS14, Section 4.3] which can be obtained from a whole-plane GFF plus $-\alpha \log |\cdot|$ by “zooming in near 0”, so as to fix the additive constant in a canonical way. In this paper we will be especially interested in the γ -quantum cone (i.e., $\alpha = \gamma$). One reason why the case when $\alpha = \gamma$ is special is that if h is some variant of the GFF on a domain $D \subset \mathbb{C}$, then near a typical point sampled from the γ -quantum measure μ_h , the field h locally looks like a whole-plane GFF plus $-\gamma \log |\cdot|$ [DS11, Proposition 3.4]. Hence a γ -quantum cone describes the behavior of a quantum surface at a quantum typical point. It is sometimes convenient to parameterize the set of quantum cones by a different parameter, called the *weight*, which is defined to be

$$\mathfrak{w} = 2\gamma(Q - \alpha). \quad (2.3)$$

We note that the γ -quantum cone has weight $4 - \gamma^2$. The reason why the weight parameter is convenient is that it is additive under the gluing and cutting operations for quantum wedges and quantum cones studied in [DMS14]. We will say more about these operations in Section 2.2.3 just below.

For $\alpha \leq Q$, an α -quantum wedge is a doubly-marked quantum surface $(\mathbb{H}, h, 0, \infty)$ defined in [DMS14, Section 4.2] which can be obtained from a free-boundary GFF on \mathbb{H} plus $-\alpha \log |\cdot|$ by “zooming in near 0”

(so as to fix the additive constant in a canonical way). Quantum wedges in the case when $\alpha \leq Q$ are called *thick wedges* because they describe surfaces homeomorphic to \mathbb{H} .

For $\alpha \in (Q, Q + \gamma/2)$, an α -quantum wedge is an ordered Poissonian collection of doubly-marked quantum surfaces, each with the topology of the disk (the two marked points correspond to the points $\pm\infty$ in the infinite strip in [DMS14, Definition 4.13]). The individual surfaces are called *beads* of the quantum wedge. One can also consider a single bead of an α -quantum wedge conditioned on its quantum area and/or its left and right quantum boundary lengths. See [DMS14, Section 4.4] for more details. Quantum wedges in the case when $\alpha \in (Q, Q + \gamma/2)$ are called *thin wedges*. The *weight* of an α -quantum wedge for $\alpha < Q + \gamma/2$ is defined by

$$\mathfrak{w} = \gamma \left(\frac{\gamma}{2} + Q - \alpha \right). \quad (2.4)$$

The main type of quantum wedge in which we will be interested in this paper is the $\frac{3\gamma}{2}$ -quantum wedge, which has weight $2 - \gamma^2/2$. This wedge is thin for $\gamma \in (\sqrt{2}, 2)$, which is the main regime in which we will work. Roughly speaking, the reason why we are interested in the $\frac{3\gamma}{2}$ -quantum wedge is that it can be obtained from a γ -quantum cone by “cutting it in half” by a pair of SLE_{γ^2} -type curves (see Section 2.3.2). A single bead of a $\frac{3\gamma}{2}$ -quantum wedge locally looks like a $(\frac{4}{\gamma} - \frac{\gamma}{2})$ -quantum wedge near the first marked point of either wedge. The latter quantum wedge is thick, and appears in [DMS14, Theorem 1.19].

It follows from the definitions in [DMS14, Section 4.4] that in the special case when $\gamma = \sqrt{8/3}$, a doubly-marked quantum disk has the same law as a single bead of a $\sqrt{6}$ -quantum wedge if we condition on quantum area and left/right quantum boundary lengths (note that this is only true for $\gamma = \sqrt{8/3}$).

2.2.3 Conformal welding and conformal removability

As alluded to in Section 2.2.2, the reason for introducing the weight parameter is that it is invariant under various cutting and gluing operations for quantum surfaces. Suppose $\mathfrak{w}^-, \mathfrak{w}^+ > 0$ and $\mathfrak{w} = \mathfrak{w}^- + \mathfrak{w}^+$. It is shown in [DMS14, Theorem 1.9] that if one cuts a weight- \mathfrak{w} quantum wedge by an independent chordal $\text{SLE}_\kappa(\mathfrak{w}^- - 2; \mathfrak{w}^+ - 2)$ curve with force points immediately to the left and right of the starting point (or a concatenation of such curves in the thin wedge case) then one obtains a weight- \mathfrak{w}^- quantum wedge and an independent- \mathfrak{w}^+ quantum wedge which can be glued together according to quantum boundary length to recover the original weight- \mathfrak{w} quantum wedge. Similarly, by [DMS14, Theorem 1.12], if one cuts a weight- \mathfrak{w} quantum cone by an independent whole-plane $\text{SLE}_\kappa(\mathfrak{w} - 2)$ curve, then one obtains a weight- \mathfrak{w} quantum wedge whose left and right boundaries can be glued together according to quantum length to recover the original weight- \mathfrak{w} quantum cone.

More generally, if we are given two quantum surfaces \mathcal{S}_1 and \mathcal{S}_2 with boundaries and an identification between their boundaries, one can attempt to conformally weld \mathcal{S}_1 and \mathcal{S}_2 along their boundaries according to this identification. In other words, one can ask if there exists a quantum surface \mathcal{S} such that \mathcal{S}_1 and \mathcal{S}_2 are sub-surfaces of \mathcal{S} whose union is all of \mathcal{S} and the identification between the boundaries of \mathcal{S}_1 and \mathcal{S}_2 induced by the inclusion maps agrees with the given identification.

If such a quantum surface \mathcal{S} exists (which will always be the case in settings we consider), one can further ask if it is unique. The surface \mathcal{S} is endowed with a distinguished subset A corresponding to the gluing interface between \mathcal{S}_1 and \mathcal{S}_2 . Uniqueness of \mathcal{S} is equivalent to the condition that this set A is *conformally removable* in \mathcal{S} , i.e. every homeomorphism from \mathcal{S} (viewed as a topological space) to itself which is conformal on $\mathcal{S} \setminus A$ is in fact conformal on all of \mathcal{S} .

The seminal paper on the topic of conformal removability is [JS00], where it is shown in particular that boundaries of Hölder domains are conformally removable in \mathbb{C} or in any simply connected sub-domain of \mathbb{C} . SLE_κ curves for $\kappa \in (0, 4)$ are boundaries of Hölder domains [RS05] so are conformally removable. It is further explained in [DMS14, Proposition 1.8] that countable unions of non-crossing SLE_κ -type curves which do accumulate only at a discrete set of points are conformally removable in \mathbb{C} or in any simply connected sub-domain of \mathbb{C} .

It is a seemingly quite difficult open problem to determine whether an SLE_κ curve is conformally removable for $\kappa \in [4, 8)$. If we knew that $\text{SLE}_{\kappa'}$ -type curves were conformally removable for $\kappa' \in (4, 8)$, the proofs in the present paper could be greatly simplified.

2.2.4 The $\sqrt{8/3}$ -LQG metric

It is shown in [MS15b, MS16b, MS16c] that in the special case when $\gamma = \sqrt{8/3}$, a $\sqrt{8/3}$ -LQG surface admits a metric in addition to its quantum area measure and quantum length measure. In particular, if $D \subset \mathbb{C}$ and h is a GFF-type distribution on D , then h induces a metric \mathfrak{d}_h on D called the $\sqrt{8/3}$ -LQG metric. This metric is constructed using a growth process called QLE(8/3, 0) which is obtained by, roughly speaking, randomly re-shuffling small increments of an SLE₆ curve in a manner which depends on h and taking a limit as the size of the increments tends to 0 (see [MS16f] for a different construction of QLE which is expected, but not yet proven, to be equivalent to the one used to define \mathfrak{d}_h in the case when $\kappa = 6$).

The metric \mathfrak{d}_h is a well-defined functional of the quantum surface (D, h) in the sense that if (D, h) and (\tilde{D}, \tilde{h}) are related by a conformal map $f : D \rightarrow \tilde{D}$ as in (2.2), then a.s. $\mathfrak{d}_h(z, w) = \mathfrak{d}_{\tilde{h}}(f(z), f(w))$ for each $z, w \in D$. In fact, it can be deduced from the results of [SW16] and the construction of \mathfrak{d}_h in [MS15b, MS16b, MS16c] that this a.s. holds for all choices of conformal map f simultaneously.

The metric \mathfrak{d}_h is a.s. determined by the field h , and conversely it is shown in [MS16c, Theorem 1.4] that the metric measure space structure of $(D, \mathfrak{d}_h, \mu_h)$ a.s. determines the quantum surface (D, h) . Hence in the special case when $\gamma = \sqrt{8/3}$, a $\sqrt{8/3}$ -LQG surface can equivalently be viewed as a random metric measure space.

Several particular types of quantum surfaces are equivalent (i.e., they differ by a measure-preserving isometry) to certain *Brownian surfaces*, random metric measure spaces which arise as the scaling limits of various types of uniform random planar maps; see Section 1.1.

2.2.5 Curves on quantum surfaces

Our main interest in this paper is in quantum surfaces decorated by various types of curves, typically SLE _{κ} - or SLE _{κ'} -type curves for $\kappa = \gamma^2$ or $\kappa' = 16/\gamma^2$ (as in (2.1)). It will be important for us to distinguish between curves in subsets of \mathbb{C} and *curve-decorated quantum surfaces*. By the latter, we mean an equivalence class of $(k+3)$ -tuples $(D, h, x_1, \dots, x_k, \eta)$ for some $k \in \mathbb{N}$, where (D, h, x_1, \dots, x_k) is an equivalence class representative for a quantum surface with k marked points and η is a curve in D , with two such $(k+3)$ -tuples $(D, h, x_1, \dots, x_k, \eta)$ and $(\tilde{D}, \tilde{h}, \tilde{x}_1, \dots, \tilde{x}_k, \tilde{\eta})$ defined to be equivalent if there is a conformal map $f : D \rightarrow \tilde{D}$ which satisfies the conditions of (2.2) and also satisfies $f \circ \eta = \tilde{\eta}$.

We allow “curves” η which are defined on the union of a disjoint union of intervals in \mathbb{R} , rather than just a single interval, which arise naturally if we want to restrict a curve to the pre-image of some set whose pre-image is not connected.

Since most of our curves will be originally defined on subsets of \mathbb{C} , we introduce the following notation.

Definition 2.1. Let (D, h, x_1, \dots, x_k) be an embedding of a quantum surface \mathcal{S} and let η be a curve in \mathbb{C} . Let I be the closure of the interior of $\eta^{-1}(\overline{D})$ (which consists of a finite collection of closed intervals) and let $\eta_{\mathcal{S}}$ to be the curve $\eta|_I$, viewed as a curve on \mathcal{S} , so that $(\mathcal{S}, \eta_{\mathcal{S}})$ is a curve-decorated quantum surface represented by the equivalence class of $(D, h, x_1, \dots, x_k, \eta)$ modulo conformal maps.

Note that $\eta_{\mathcal{S}}$ does not encode times when η bounces off the boundary of ∂D without spending a positive interval of time in \overline{D} (this is because we restrict to I instead of $\eta^{-1}(\overline{D})$).

2.2.6 Convergence of quantum surfaces

In this subsection, we will describe how to define a topology on certain families of quantum surfaces (possibly decorated by curves) using the perspective that a quantum surface is the same as an equivalence class of measure spaces modulo conformal maps.

We first define a metric on certain curve-decorated measure spaces viewed modulo conformal maps, which leads to a notion of convergence for simply connected, possibly curve-decorated, quantum surfaces. For $k \in \mathbb{N}$, let $\mathbb{M}_k^{\text{CPU}}$ be the set of all equivalence classes \mathcal{K} of $(k+3)$ -tuples $(D, \mu, \eta, x_1, \dots, x_k)$ with $D \subset \mathbb{C}$ a simply connected domain, μ a finite Borel measure on D which is finite on compact subsets of \mathbb{D} , $\eta : \mathbb{R} \rightarrow \overline{D}$ a curve which extends continuously to the extended real line $\mathbb{R} \cup \{-\infty, \infty\}$, $x_1 \in D$, and $x_2, \dots, x_k \in D \cup \partial D$ (with ∂D viewed as a collection of prime ends). Two such $k+3$ -tuples $(D, \mu, \eta, x_1, \dots, x_k)$ and $(\tilde{D}, \tilde{\mu}, \tilde{\eta}, \tilde{x}_1, \dots, \tilde{x}_k)$ are declared to be equivalent if there exists a conformal map $f : D \rightarrow \tilde{D}$ such that

$$f_*\mu = \tilde{\mu}, \quad f \circ \eta = \tilde{\eta}, \quad \text{and} \quad f(x_j) = \tilde{x}_j, \quad \forall j \in [1, k]_{\mathbb{Z}}. \quad (2.5)$$

The reason why x_1 is required to be in D (not in ∂D) is as follows. Just below, we will define a metric on $\mathbb{M}_k^{\text{CPU}}$ by “embedding” two elements of $\mathbb{M}_k^{\text{CPU}}$ into the unit disk in such a way that their first marked points are sent to 0. If we allows arbitrary embeddings, our metric would not be positive definite due to conformal maps which carry all of the mass to the boundary.

If $(D, \mu, x_1, \dots, x_k)$ is a finite measure space (without a curve) with k marked points, the first of which is in the interior, viewed modulo conformal maps, then $(D, \mu, x_1, \dots, x_k)$ can be viewed as an element of $\mathbb{M}_k^{\text{CPU}}$ whose corresponding curve is constant at x_1 .

The discussion in Section 2.2.1 implies that a finite-area curve-decorated γ -quantum surface $(D, h, \eta, x_1, \dots, x_k)$ with k marked points such that $x_1 \in D$ can be viewed as an element of $\mathbb{M}_k^{\text{CPU}}$, with $\mu = \mu_h$ the γ -quantum area measure. A quantum surface without an interior marked point can be made into a quantum surface with an interior marked point by, e.g., sampling a point uniformly from $\mu_h|_U$ (normalized to be a probability measure) for some open set $U \subset D$. We can thereby view such a curve-decorated quantum surface as an element of $\mathbb{M}_{k+1}^{\text{CPU}}$. Curve-decorated quantum surfaces will be our main examples of elements of $\mathbb{M}_k^{\text{CPU}}$.

To define a metric on $\mathbb{M}_k^{\text{CPU}}$, we note that the Riemann mapping theorem implies that each $\mathcal{K} \in \mathbb{M}_k^{\text{CPU}}$ admits an embedding of the form $(\mathbb{D}, \mu, \eta, 0, x_1, \dots, x_k)$ (i.e., the domain is \mathbb{D} and the first marked point is 0). For a domain $D \subset \mathbb{C}$, let $\mathfrak{d}_D^{\text{P}}$ be the Prokhorov metric on finite Borel measures on D and let $\mathfrak{d}_D^{\text{U}}$ be the uniform metric on curves in D . We define the *conformal Prokhorov-uniform distance* between elements $\mathcal{K}_1, \tilde{\mathcal{K}} \in \mathbb{M}_k^{\text{CPU}}$ by the formula

$$\mathfrak{d}_k^{\text{CPU}}(\mathcal{K}, \tilde{\mathcal{K}}) = \inf_{\substack{(\mathbb{D}, \mu, \eta, 0, \dots, x_k) \in \mathcal{K}, \\ (\mathbb{D}, \tilde{\mu}, \tilde{\eta}, 0, \dots, \tilde{x}_k) \in \tilde{\mathcal{K}}}} \left\{ \mathfrak{d}_{\mathbb{D}}^{\text{P}}(\mu, \tilde{\mu}) + \mathfrak{d}_{\mathbb{D}}^{\text{U}}(\eta, \tilde{\eta}) + \sum_{j=2}^k |x_j - \tilde{x}_j| \right\}. \quad (2.6)$$

It is easily verified that $\mathfrak{d}_k^{\text{CPU}}$ is a metric on $\mathbb{M}_k^{\text{CPU}}$, whereby \mathcal{K} and $\tilde{\mathcal{K}}$ are $\mathfrak{d}_k^{\text{CPU}}$ -close if they can be embedded into \mathbb{D} in such a way that their first marked points are sent to 0, their measures are close in the Prokhorov distance, their curves are close in the uniform distance, and their corresponding marked points are close in the Euclidean distance.

The above construction gives us a topology on simply connected curve-decorated quantum surfaces. We will also have occasion to consider convergence of beaded quantum surfaces, i.e. those which can be represented as a countable ordered collection of finite quantum surfaces, with each such surface attached to its neighbors at a pair of points, such that quantum area of the beads is locally finite, i.e., the total quantum area of the beads between any two give beads is finite. Examples of beaded quantum surfaces include thin quantum wedges.

Suppose $\mathcal{K} = (\mathcal{S}, \eta_{\mathcal{S}})$ is a curve-decorated beaded quantum surface with the property that $\eta_{\mathcal{S}}$ enters the beads of \mathcal{S} in chronological order and does not re-enter any bead after entering a subsequent bead. Let $T \in (0, \infty]$ be the total mass of the beads in \mathcal{S} and suppose that each bead in \mathcal{S} has k marked points, the first of which is in the interior. We can view \mathcal{K} as a function $[0, T] \rightarrow \mathbb{M}_k^{\text{CPU}}$ which is defined at Lebesgue a.e. point of $[0, T]$ as follows. For $t \in [0, T]$, we define \mathcal{K}_t to be the curve-decorated quantum surface consisting of the first bead of \mathcal{S} with the property that the sum of the quantum masses of the previous beads (not including the bead itself) is at least t , equipped with the segment of $\eta_{\mathcal{S}}$ which is contained in this bead. We extend this function to all of $[0, \infty)$ by declaring that \mathcal{K}_t is the trivial element of $\mathbb{M}_k^{\text{CPU}}$ (i.e., the one whose total mass is 0, whose marked points all coincide, and whose curve is constant at the marked point) for each $t > T$.

We define $\mathbb{M}_k^{\text{bead}}$ to be the set of all Borel measurable functions $\mathcal{K} : [0, \infty) \rightarrow \mathbb{M}_k^{\text{CPU}}$ which are defined a.e., so that a beaded curve-decorated quantum surface is an element of $\mathbb{M}_k^{\text{bead}}$ in the manner described above. We then define a metric on $\mathbb{M}_k^{\text{bead}}$ by

$$\mathfrak{d}_k^{\text{bead}}(\mathcal{K}, \tilde{\mathcal{K}}) = \int_0^\infty e^{-t} \left(1 \wedge \mathfrak{d}_k^{\text{CPU}}(\mathcal{K}_t, \tilde{\mathcal{K}}_t) \right) dt. \quad (2.7)$$

As in the case of simply connected quantum surfaces, a beaded quantum surface without a curve can be viewed as an element of $\mathbb{M}_k^{\text{bead}}$ by declaring the curve corresponding to each \mathcal{K}_t to be constant at the first marked point.

2.3 Space-filling $\text{SLE}_{\kappa'}$ and the peanosphere construction

We will now review the definition of space-filling $\text{SLE}_{\kappa'}$ and its relationship to various γ -quantum surfaces. We continue to assume that γ , κ , and κ' are related as in (2.1).

2.3.1 Imaginary geometry

Let

$$\chi = \frac{2}{\sqrt{\kappa}} - \frac{\sqrt{\kappa}}{2}. \quad (2.8)$$

Suppose D is a simply connected domain in \mathbb{C} and h is a GFF on D with piecewise constant boundary data which changes only finitely many times. The work [MS16d] studies various couplings of h with certain chordal $\text{SLE}_{\kappa}(\rho)$ and $\text{SLE}_{\kappa'}(\rho)$ curves between points of ∂D which are called *flow lines* and *counterflow lines* of h , respectively. These curves are a.s. determined by h and are local sets for h in the sense of [SS13]. One can consider flow lines with any angle θ in a certain range depending on the boundary data of h , which gives rise to couplings of multiple $\text{SLE}_{\kappa}(\rho)$ curves started from the same point with the same GFF. Flow lines with the same angle started at different points a.s. merge upon intersecting, whereas flow lines with different angles can cross at most once (depending on the angles).

It is shown in [MS13] that one can also make sense of flow lines of h started from points in the interior of D . These curves are not exactly $\text{SLE}_{\kappa}(\rho)$ curves but locally look like such curves. Furthermore, suppose h is a whole-plane GFF, viewed modulo a global additive multiple of $2\pi\chi$ (with χ as in (2.8)). Then one can make sense of flow lines of h started from any given point in \mathbb{C} as well as counterflow lines of h started from ∞ and targeted at any given point in \mathbb{C} . The flow lines of h for any angle in $[-\pi, \pi)$ are whole-plane $\text{SLE}_{\kappa}(2-\kappa)$ curves [MS13, Theorem 1.1] and the counterflow lines are whole-plane $\text{SLE}_{\kappa'}(\kappa'-6)$ curves [MS13, Theorem 1.6]. By [MS13, Theorem 1.15], for $z \in \mathbb{C}$ it is a.s. the case that the flow lines of h started from z with angles $\pm\pi/2$ are the left and right outer boundaries of the counterflow line of h from ∞ to z if we lift this counterflow line to a path in the universal cover of \mathbb{C} .

We note that these two flow lines will a.s. intersect each other infinitely often if $\kappa \in (2, 4)$, but not if $\kappa \leq 2$ [MS13, Theorem 1.11].

2.3.2 Space-filling $\text{SLE}_{\kappa'}$

In this subsection we review the construction of space-filling $\text{SLE}_{\kappa'}$ from [MS13, Sections 1.2.3 and 4.3] and [DMS14, Footnote 9]. Suppose that h is a whole-plane GFF viewed modulo a global additive multiple of $2\pi\chi$, with χ as in (2.8). Let \mathcal{Q} be a countable dense subset of \mathbb{C} and for $z \in \mathcal{Q}$, let η_z^- and η_z^+ be the flow lines of h started from z with angles $\pi/2$ and $-\pi/2$, respectively (recall Section 2.3.1).

We define a total order on \mathcal{Q} by declaring that z comes before w if and only if z lies in a connected component of $\mathbb{C} \setminus (\eta_w^- \cup \eta_w^+)$ whose boundary is traced by the left side of η_w^- and the right side of η_w^+ . It follows from the argument of [MS13, Section 4.3] that there is a space-filling curve η' from ∞ to ∞ in \mathbb{C} which hits points in \mathcal{Q} in order and is continuous when parameterized by Lebesgue measure. Furthermore, the law of this curve does not depend on the choice of countable dense subset \mathcal{Q} . This curve is called *whole-plane space-filling $\text{SLE}_{\kappa'}$ from ∞ to ∞* .

For $\kappa' \geq 8$, whole-plane space-filling $\text{SLE}_{\kappa'}$ is just a two-sided variant of ordinary $\text{SLE}_{\kappa'}$. For $\kappa' \in (4, 8)$, the space-filling $\text{SLE}_{\kappa'}$ curve η' evolves in a similar manner to an $\text{SLE}_{\kappa'}$ -type curve, but whenever it hits itself and forms a bubble, it enters the bubble and fills it in with a space-filling loop rather than just continuing outside the bubble. In particular, whole-plane space-filling $\text{SLE}_{\kappa'}$ from ∞ to ∞ in this case is not described by a Loewner evolution, even locally. The path targeted at a given point $z \in \mathbb{C}$ (i.e., parameterized by capacity as seen from z) has the law of an $\text{SLE}_{\kappa'}(\kappa'-6)$ process and is in fact that the counterflow line of h from ∞ to z . That is, this counterflow line can be recovered from η' by skipping all of the bubbles filled in by η' before it hits z .

One can perform a similar construction to the above starting from a GFF on a proper simply connected sub-domain of \mathbb{C} with appropriate boundary data, rather than a GFF on \mathbb{C} (this construction which is described explicitly in [MS13, Sections 1.2.3 and 4.3]). This gives rise to chordal space-filling $\text{SLE}_{\kappa'}(\rho^L; \rho^R)$ processes for $\rho^L, \rho^R \in (-2, \kappa'/2 - 2)$. For $\kappa' \geq 8$, chordal space-filling $\text{SLE}_{\kappa'}(\rho^L; \rho^R)$ is identical to ordinary

chordal $\text{SLE}_{\kappa'}(\rho^L; \rho^R)$, and for $\kappa' \in (4, 8)$ it is obtained from chordal $\text{SLE}_{\kappa'}(\rho^L; \rho^R)$ by iteratively filling in the bubbles it disconnects from its target point.

As explained in [DMS14, Footnote 9], whole-plane space-filling $\text{SLE}_{\kappa'}$ from ∞ to ∞ can equivalently be constructed from chordal $\text{SLE}_{\kappa'}$, as follows. Let h is a whole-plane GFF viewed modulo a global additive multiple of $2\pi\chi$, as above, and let η_0^- and η_0^+ be the flow lines of h started from 0 with angles $\pi/2$ and $-\pi/2$, respectively. Conditional on η_0^- and η_0^+ , sample an independent chordal space-filling $\text{SLE}_{\kappa'}$ in each connected component U of $\mathbb{C} \setminus (\eta_0^- \cup \eta_0^+)$, between the two points of ∂U where η_0^- and η_0^+ intersect (or between 0 and ∞ , if $\kappa' \geq 8$ in which case η_0^\pm do not intersect). Then concatenate these chordal space-filling $\text{SLE}_{\kappa'}$ curves.

2.3.3 Peanosphere construction

It is particularly natural to consider a γ -quantum cone (recall Section 2.2.2) decorated by an independent whole-plane space-filling $\text{SLE}_{\kappa'}$ curve from ∞ to ∞ . The reason for this is the so-called *peanosphere construction*, which we now describe.

Let $\mathcal{C} = (\mathbb{C}, h, 0, \infty)$ be a γ -quantum cone and let η' be a whole-plane space-filling $\text{SLE}_{\kappa'}$ curve from ∞ to ∞ independent from h and parameterized in such a way that $\eta'(0) = 0$ and the γ -quantum area measure satisfies $\mu_h(\eta'([s, t])) = t - s$ whenever $s, t \in \mathbb{R}$ with $s < t$. For $t \geq 0$, let L_t be equal to the γ -quantum length of the segment of the left boundary of $\eta'([t, \infty))$ which is shared with $\eta'([0, t])$ minus the γ -quantum length of the segment of the left boundary of $\eta'([0, t])$ which is not shared with $\eta'([t, \infty))$; and for $t < 0$, let L_t be the γ -quantum length of the segment of the left boundary of $\eta'([t, 0])$ which is not shared with $\eta'((-\infty, t])$ minus the γ -quantum length of the segment of the left boundary of $\eta'([t, 0])$ which is shared with $\eta'((-\infty, t])$. Define R_t similarly with “right” in place of “left”. Also let $Z := (L, R) : \mathbb{R} \rightarrow \mathbb{R}^2$.

It is shown in [DMS14, Theorem 1.13] (see also [GHMS17] for the case $\kappa' \geq 8$) that there is a deterministic constant $\alpha = \alpha(\gamma) > 0$ such that Z evolves as a pair of correlated two-dimensional Brownian motions with variances and covariances given by (1.2).

The Brownian motion Z is referred to as the *peanosphere Brownian motion*. The reason for the name is that Z can be used to construct a random curve-decorated topological space called an *infinite-volume peanosphere*, which a.s. differs from (\mathbb{C}, η') by a curve-preserving homeomorphism. See Figure 2 for an illustration. We remark that there is also a finite-volume analog of the peanosphere construction, with a quantum sphere in place of a γ -quantum cone and a pair of correlated Brownian excursions in place of a pair of correlated Brownian motions. See [MS15c] for more details.

By [DMS14, Theorem 1.14], Z a.s. determines the curve-decorated quantum surface $(\mathcal{C}, \eta'_\mathcal{C})$. This determination is local, in the following sense. For $a, b \in \mathbb{R} \cup \{-\infty, \infty\}$ with $a < b$, let $\mathcal{S}_{a,b}$ be the beaded quantum surface parameterized by the interior of $\eta'([a, b])$. Then the curve-decorated quantum surface $(\mathcal{S}_{a,b}, \eta'_{\mathcal{S}_{a,b}})$, but not its particular embedding into \mathbb{C} , is a.s. determined by $(Z - Z_a)|_{[a,b]}$ (see [GHS16b, Lemma 3.12] for a careful justification of this point). Furthermore, it follows from [DMS14, Theorems 1.9 and 1.12] (c.f. the proof of [DMS14, Lemma 9.2]) that for $t \in \mathbb{R}$, the quantum surfaces $\mathcal{S}_{-\infty, t}$ and $\mathcal{S}_{t, \infty}$ are $\frac{3\gamma^2}{2}$ -quantum wedges and the curve-decorated quantum surfaces $(\mathcal{S}_{-\infty, t}, \eta'_{\mathcal{S}_{-\infty, t}})$ and $(\mathcal{S}_{t, \infty}, \eta'_{\mathcal{S}_{t, \infty}})$ are independent (here we use Definition 2.1). In particular, the hypotheses of Theorem 1.1 are satisfied in the case when $(\tilde{h}, \tilde{\eta}') = (h, \eta')$ is an embedding into $(\mathbb{C}, 0, \infty)$ of a γ -quantum cone together with an independent whole-plane space-filling $\text{SLE}_{\kappa'}$.

2.3.4 Describing events in terms of the peanosphere Brownian motion

Many objects associated with the pair (h, η') can be described explicitly in terms of the peanosphere Brownian motion Z . In this paper the following will be particularly relevant. We note that the second and third sets mentioned below are empty for $\kappa' \geq 8$.

- The curve η' hits the left (resp. right) outer boundary of $\eta'((-\infty, t])$ at a time $s \geq t$ without forming a bubble if and only if L (resp. R) attains a running infimum at time s relative to time t .
- The points where the left and right outer boundaries of $\eta'([t, \infty))$ intersect are precisely the times $s \geq t$ at which L and R attain a simultaneous running infimum.
- The set of bubbles filled in by η' are precisely the sets of the form $\eta'([v_Z(t), t])$ for t a $\pi/2$ -cone time for Z , defined just below. Furthermore, the quantum area (resp. quantum boundary length) of the bubble

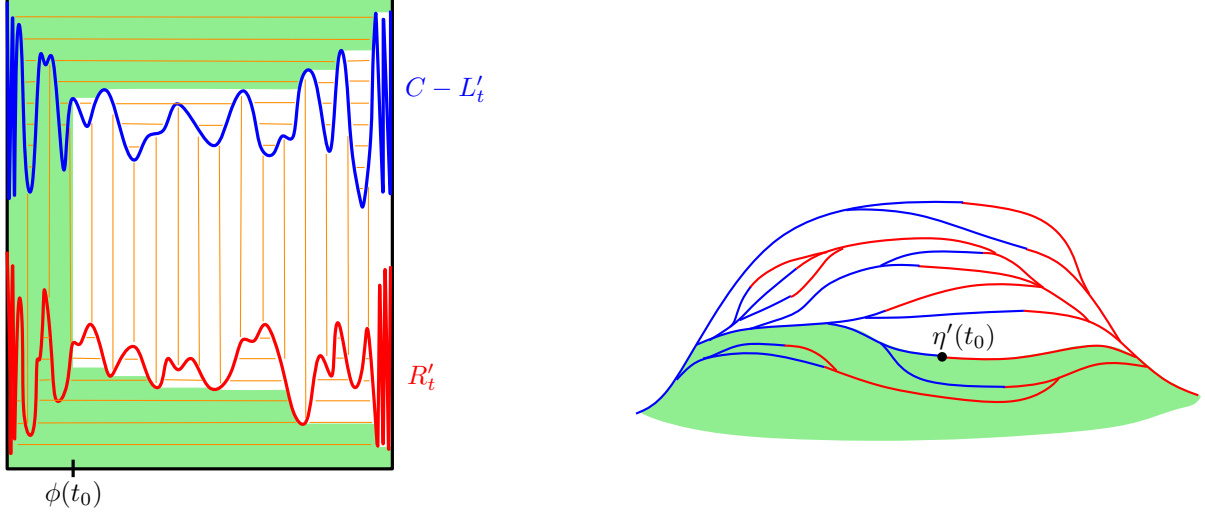


Figure 2: An illustration of the definition of the peanosphere as a curve-decorated topological space, which first appeared in [GHS16b]. Let $Z = (L_t, R_t)_{t \in \mathbb{R}}$ be a correlated two-dimensional Brownian motion as in (1.2). Let $\phi : \mathbb{R} \rightarrow (0, 1)$ be an increasing, continuous and bijective function, and for $t \in (0, 1)$ define $L'_t := \phi(L_{\phi^{-1}(t)})$ and $R'_t := \phi(R_{\phi^{-1}(t)})$. The left figure shows R' and $C - L'$, where C is a constant chosen so large that the two graphs do not intersect. We draw horizontal lines above the graph of $C - L'$ and below the graph of R' , in addition to vertical lines between the two graphs, and then we identify points which lie on the same horizontal or vertical line segment. By Moore's theorem [Moo28] the resulting quotient space S is a topological sphere. This sphere is decorated with a space-filling path η' where $\eta'(t)$ for $t \in \mathbb{R}$ is the equivalence class of $\phi(t) \in (0, 1)$. We call the pair (S, η') an *infinite-volume peanosphere*. It is shown in [DMS14] that a γ -quantum cone decorated by an independent whole-plane space-filling $\text{SLE}_{\kappa'}$ parameterized by γ -quantum mass is a.s. topologically equivalent to the infinite-volume peanosphere constructed from its peanosphere Brownian motion Z .

$\eta'([v_Z(t), t])$ is equal to $t - v_Z(t)$ (resp. $|Z_t - Z_{v_Z(t)}|$) and the boundary of $\eta'([v_Z(t), t])$ is traversed by the left (resp. right) side of η' if and only if $R_t = R_{v_Z(t)}$ (resp. $L_t = L_{v_Z(t)}$).

Definition 2.2. A time $t \in [a, b]$ is called a $\pi/2$ -cone time for a function $Z = (L, R) : \mathbb{R} \rightarrow \mathbb{R}^2$ if there exists $t' < t$ such that $L_s \geq L_t$ and $R_s \geq R_t$ for each $s \in [t', t]$. Equivalently, $Z([t', t])$ is contained in the cone $Z(t) + \{z \in \mathbb{C} : \arg z \in [0, \pi/2]\}$. We write $v_Z(t)$ for the infimum of the times t' for which this condition is satisfied, i.e. $v_Z(t)$ is the entrance time of the cone. The $\pi/2$ -cone interval corresponding to the time t is $[v_Z(t), t]$ and the corresponding $\pi/2$ -cone excursion is $(Z - Z_{v_Z(t)})|_{[v_Z(t), t]}$.

See Figure 3 for an illustration of Definition 2.2. A positively correlated Brownian motion (which corresponds to $\kappa' \in (4, 8)$) a.s. has an uncountable fractal set of $\pi/2$ -cone times, whereas an uncorrelated or negatively correlated Brownian motion (corresponding to $\kappa' \geq 8$) a.s. has no $\pi/2$ -cone times [Shi85, Eva85]. We note that the right endpoint of a $\pi/2$ -cone interval containing t is the same as a simultaneous running infimum for L and R relative to time t .

There are certain special $\pi/2$ -cone times (corresponding to special bubbles filled in by η') which will be especially important in this paper.

Definition 2.3. A $\pi/2$ -cone time for Z is called a *maximal $\pi/2$ -cone time* in an (open or closed) interval $I \subset \mathbb{R}$ if $[v_Z(t), t] \subset I$ and there is no $\pi/2$ -cone time t' for Z such that $[v_Z(t'), t'] \subset I$ and $[v_Z(t), t] \subset (v_Z(t'), t')$. In this case the interval $[v_Z(t), t]$ is called a *maximal $\pi/2$ -cone interval* for Z in I .

2.4 Stronger characterization theorem for whole-plane space-filling $\text{SLE}_{\kappa'}$

Here we state a stronger version of the characterization result Theorem 1.1 with weaker but more complicated hypotheses. This is the version of the theorem which we will actually prove, and the version which will be

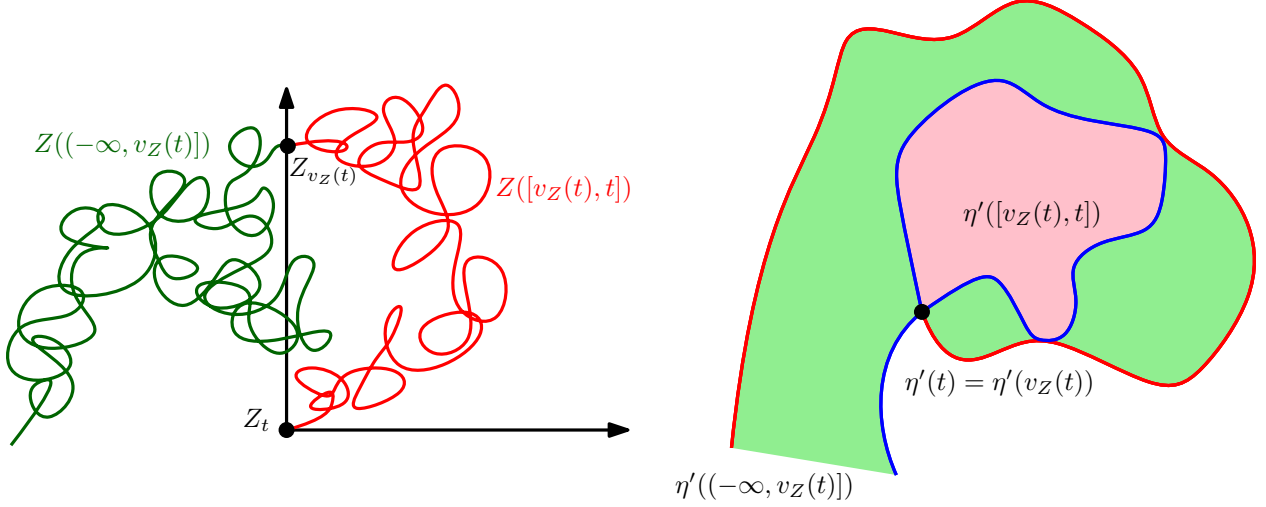


Figure 3: **Left:** A $\pi/2$ -cone time t for Z along with the corresponding cone entrance time $v_Z(t)$. **Right:** The corresponding behavior of the space-filling $\text{SLE}_{\kappa'}$ curve η' . At time $v_Z(t)$, η' closes off a bubble which it fills in during the time interval $[v_Z(t), t]$.

used to deduce our other characterization theorems in Sections 6 and 7.

Before stating the theorem, we introduce some notation which will be used to formulate the Markov property condition. Let $(\tilde{h}, \tilde{\eta}', Z)$ be a coupling of an embedding into \mathbb{C} of a γ -quantum cone, a space-filling curve in $\mathbb{R} \rightarrow \mathbb{C}$, and a correlated two-dimensional Brownian motion with variances and covariance as in (1.2), as in Theorem 1.1. For $a, b \in \mathbb{R} \cup \{-\infty, \infty\}$ with $a < b$, let $\tilde{\mathcal{F}}_{a,b}$ be the σ -algebra generated by $(Z - Z_a)|_{[a,b]}$ (or $(Z - Z_b)|_{[a,b]}$ if $a = -\infty$) and the singly marked quantum surfaces $(\tilde{\eta}'([v_Z(s), s]), \tilde{h}|_{\tilde{\eta}'([v_Z(s), s])}, \tilde{\eta}'(s))$, where s ranges over all $\pi/2$ -cone times for Z which are maximal in some interval contained in (a, b) with rational endpoints (see Section 2.3.4 for the definition of a $\pi/2$ -cone time). We note that the quantum surfaces parameterized by $\pi/2$ -cone times used in the σ -algebras $\tilde{\mathcal{F}}_{a,b}$ correspond to the “bubbles” filled in by $\tilde{\eta}'$ during the time interval $[a, b]$; c.f. Section 2.3.4.

Theorem 2.4 (Whole-plane space-filling SLE characterization). *Let $\kappa' \in (4, 8)$ and $\gamma = 4/\sqrt{\kappa'} \in (\sqrt{2}, 2)$. Suppose that $(\tilde{h}, \tilde{\eta}', Z)$ is a coupling where \tilde{h} is an embedding into $(\mathbb{C}, 0, \infty)$ of a γ -quantum cone, $\tilde{\eta}' : \mathbb{R} \rightarrow \mathbb{C}$ is a random continuous curve parameterized by γ -quantum mass with respect to \tilde{h} with $\tilde{\eta}'(0) = 0$, and Z is a correlated two-dimensional Brownian motion with variances and covariance as in (1.2). Assume that the following conditions are satisfied.*

1. (Markov property) *For each $t \in \mathbb{R}$, the σ -algebras $\tilde{\mathcal{F}}_{-\infty, t}$ and $\tilde{\mathcal{F}}_{t, \infty}$ defined just above are independent and the doubly marked beaded quantum surface $(\eta'([t, \infty)), \tilde{h}|_{\tilde{\eta}'([t, \infty))}, \tilde{\eta}'(t), \infty)$ has the law of a $\frac{3\gamma}{2}$ -quantum wedge and is independent from $\tilde{\mathcal{F}}_{-\infty, t}$.*
2. (Topology and consistency) *The curve-decorated topological space $(\mathbb{C}, \tilde{\eta}')$ is equivalent to the infinite-volume peanosphere generated by Z . Equivalently, if $((\mathbb{C}, h, 0, \infty), \eta')$ is the pair consisting of a γ -quantum cone and an independent space-filling $\text{SLE}_{\kappa'}$ from ∞ to ∞ parameterized by γ -quantum mass with respect to h which is determined by Z via [DMS14, Theorem 1.14], then there is a homeomorphism $\Phi : \mathbb{C} \rightarrow \mathbb{C}$ with $\Phi \circ \eta' = \tilde{\eta}'$. Moreover, Φ a.s. pushes forward the γ -quantum length measure on $\partial\eta'([t, \infty))$ with respect to h to the γ -quantum length measure on $\partial\tilde{\eta}'([t, \infty))$ with respect to \tilde{h} for each $t \in \mathbb{Q}$.*

Then $(\tilde{h}, \tilde{\eta}')$ is an embedding into $(\mathbb{C}, 0, \infty)$ of a γ -quantum cone together with an independent whole-plane space-filling $\text{SLE}_{\kappa'}$ from ∞ to ∞ parameterized by γ -quantum mass with respect to \tilde{h} . In fact, the map Φ of condition 2 is a.s. given by multiplication by a complex number.

As explained after the statement of Theorem 1.1, in the setting of Theorem 2.4 the γ -quantum length measure on $\partial\tilde{\eta}'([t, \infty))$ with respect to \tilde{h} is well-defined for all times $t \in \mathbb{R}$ simultaneously since we know that the law of the future unexplored quantum surface is a $\frac{3\gamma}{2}$ -quantum wedge.

Theorem 2.4 is strictly stronger than Theorem 1.1. The topology and consistency hypothesis, namely condition 2 of Theorem 2.4, is identical to condition 2 in Theorem 1.1. Both theorems also still make the assumption that the future quantum surface has the law of a $\frac{3\gamma}{2}$ -quantum wedge. However, the quantum domain Markov property assumed in Theorem 2.4 is weaker than the analogous Markov property assumed in Theorem 1.1 in two respects.

- The information about the “past” which we consider in Theorem 2.4 is encoded by the σ -algebra $\tilde{\mathcal{F}}_{-\infty, t}$, which is contained in the σ -algebra generated by the curve-decorated quantum surface $(\tilde{\eta}'((-\infty, t]), \tilde{h}|_{\tilde{\eta}'((-\infty, t])}, \tilde{\eta}'|_{(-\infty, t]})$; whereas in Theorem 2.4 we consider the entire past curve-decorated quantum surface $(\tilde{\eta}'((-\infty, t]), \tilde{h}|_{\tilde{\eta}'((-\infty, t])}, \tilde{\eta}'|_{(-\infty, t]})$.
- The Markov property in Theorem 2.4 is split into two parts: instead of assuming independence of the entire past and future curve-decorated quantum surfaces, we assume only that each of $\tilde{\mathcal{F}}_{t, \infty}$ and $(\eta'([t, \infty)), \tilde{h}|_{\eta'([t, \infty))}, \eta'(t), \infty)$ (which are each determined by the future curve-decorated quantum surface) is independent from $\tilde{\mathcal{F}}_{-\infty, t}$. Note that we do *not* assume that these two objects are jointly independent from $\tilde{\mathcal{F}}_{-\infty, t}$.

We remark that for a space-filling $\text{SLE}_{\kappa'}$ on an independent γ -quantum cone, for $a < b$ the peanosphere Brownian motion increment $(Z - Z_a)|_{[a, b]}$ a.s. determines the corresponding curve-decorated quantum surface $(\eta'([a, b]), h|_{\eta'([a, b])}, \eta'(a), \eta'(b))$ but the analogous statement for the given triple $(\tilde{h}, \tilde{\eta}', Z)$ in Theorem 2.4 is not known a priori. In particular, it is a priori possible that $\tilde{\mathcal{F}}_{-\infty, t}$ is a strict subset of $\sigma(\tilde{\eta}'((-\infty, t]), \tilde{h}|_{\tilde{\eta}'((-\infty, t])}, \tilde{\eta}'|_{(-\infty, t]})$.

Sections 3 through 5 will be devoted to the proof of Theorem 2.4.

3 Laws of surfaces and curves

Throughout this section we assume we are in the setting of Theorem 2.4, so in particular $(\tilde{h}, \tilde{\eta}')$ is our given field-curve pair; (h, η') is an embedding into $(\mathbb{C}, 0, \infty)$ of a γ -quantum cone decorated by an independent space-filling $\text{SLE}_{\kappa'}$; and $Z = (L, R)$ is the peanosphere Brownian motion for the pair (h, η') . In Section 3.1, we will define several objects associated with the pairs (h, η') and $(\tilde{h}, \tilde{\eta}')$ which will be used throughout the remainder of the paper. These include the surface $\mathcal{S}_{a, b}$ parameterized by $\eta'([a, b])$, the chordal $\text{SLE}_{\kappa'}$ -type curve $\eta_{a, b}$ contained in $\eta'([a, b])$, the surface $\mathcal{S}_{a, b}^0$ parameterized by the bubbles cut out by $\eta_{a, b}$, the function $P_{t, \infty}$ which encodes the quantum areas and left/right quantum boundary lengths of the beads of $\mathcal{S}_{t, \infty}$, and the analogs of these objects with $(\tilde{h}, \tilde{\eta}')$ in place of (h, η') .

In the remaining subsections, we study the laws of the above objects. In Section 3.2, we will prove some measurability statements for the objects introduced in Section 3.1, which in particular imply that the surface $\mathcal{S}_{a, b}$ is a.s. determined by the structure of the bubbles cut out by the curve $\eta_{a, b}$, in analogy with the measurability statements in [DMS14, Theorems 1.17 and 1.18]. In Section 3.3, we show that the bubbles of the surface $\mathcal{S}_{a, b}^0$ cut out by $\eta_{a, b}$ are independent quantum disks conditional on area and boundary length. In Section 3.4, we show that the same is true for the analogs of these surfaces with $(\tilde{h}, \tilde{\eta}')$ in place of (h, η') . In Section 3.5, we prove equality of the joint laws of certain collections of quantum surfaces defined in terms of (h, η') and their analogs defined in terms of $(\tilde{h}, \tilde{\eta}')$ which will be important in Section 5.

Throughout, we recall the notation for curves on quantum surfaces from Definition 2.1: if h is an embedding of a quantum surface \mathcal{S} into a domain $D \subset \mathbb{C}$ and η' is a curve in \mathbb{C} , we write $\eta'_\mathcal{S}$ for the curve $\eta'|_{(\eta')^{-1}(\overline{D})}$, viewed as a curve on \mathcal{S} .

3.1 Definitions of surfaces and curves

In this subsection we will define several objects associated with the pairs (h, η') and $(\tilde{h}, \tilde{\eta}')$ which we will use throughout the remainder of the paper. See Figure 4 for an illustration.

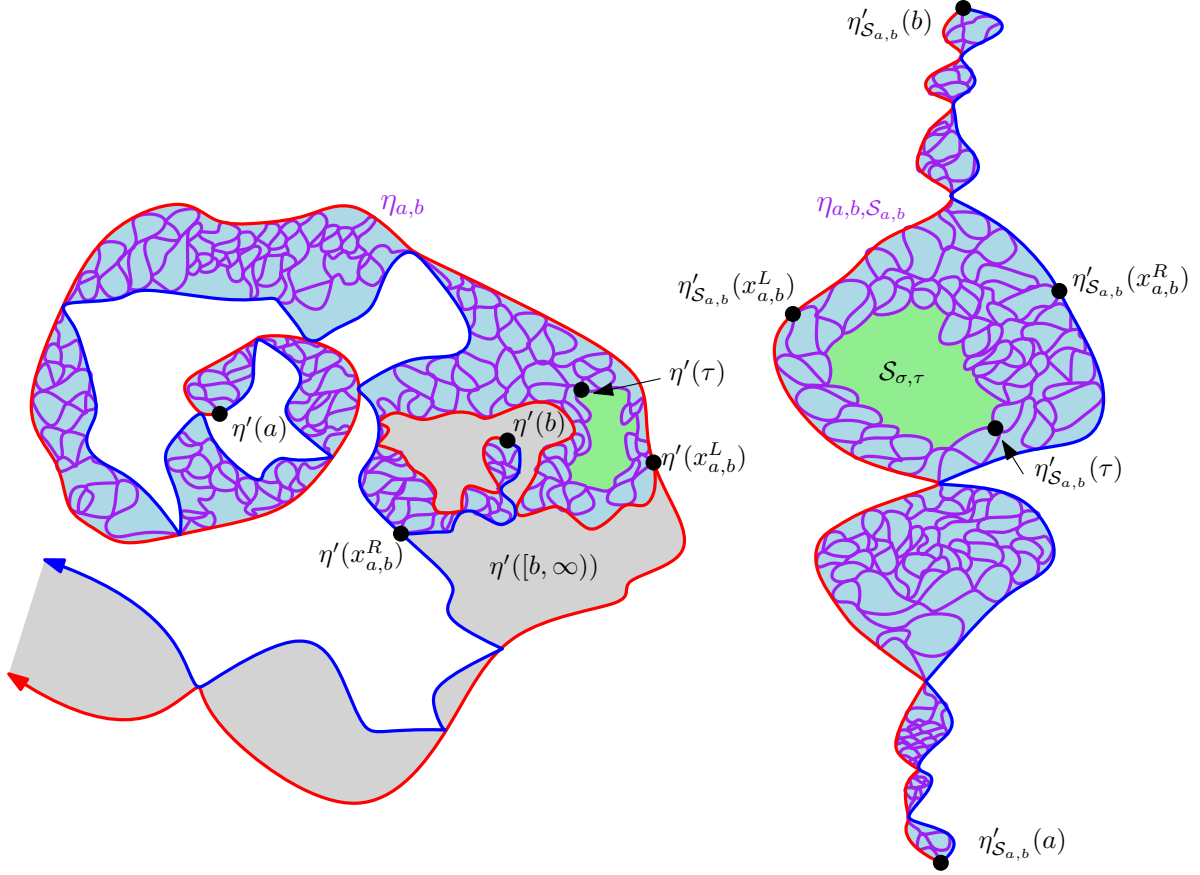


Figure 4: **Left:** The curve segments $\eta'([a, b])$ (light blue) and the curve segment $\eta'([b, \infty))$ (grey) together with the chordal $\text{SLE}_{\kappa'}$ -type curve $\eta_{a,b} : [a, b] \rightarrow \eta'([a, b])$ (purple) obtained by skipping the bubbles filled in by η' during the time interval $[a, b]$. One of these bubbles is shown in light green. This bubble is equal to the set $\eta'([\sigma, \tau])$ where $[\sigma, \tau] = [\sigma_{a,b}(t), \tau_{a,b}(t)]$ is the maximal $\frac{\pi}{2}$ -cone interval for Z in $[a, b]$ which contains the time $t \in [a, b]$. The marked points $\eta'(x_{a,b}^L)$ and $\eta'(x_{a,b}^R)$, respectively, are the points where the left and right outer boundaries, respectively, of $\eta'((-\infty, a])$ and $\eta'([b, \infty))$ meet. **Right:** The beaded quantum surface $\mathcal{S}_{a,b}$ parameterized by $\eta'([a, b])$. Note that $\mathcal{S}_{a,b}$ does not encode the exterior self-intersections of $\eta'([a, b])$; one can think of this surface as being obtained by “unwinding” $\eta'([a, b])$. The curve $\eta_{a,b,\mathcal{S}_{a,b}}$ on this surface is shown in purple. The surface $\mathcal{S}_{a,b}^0$ is parameterized by the set of bubbles cut out by $\eta_{a,b,\mathcal{S}_{a,b}}$, i.e. the region $\mathcal{S}_{a,b} \setminus \eta_{a,b,\mathcal{S}_{a,b}}$. The bubble $\eta'([\sigma, \tau])$ parameterizes the quantum surface $\mathcal{S}_{\sigma,\tau}$ (which we will eventually show is a quantum disk), which is a sub-surface of $\mathcal{S}_{a,b}^0$.

Define the γ -quantum cones

$$\mathcal{C} := (\mathbb{C}, h, 0, \infty) \quad \text{and} \quad \tilde{\mathcal{C}} := (\mathbb{C}, \tilde{h}, 0, \infty) \quad (3.1)$$

For $a, b \in \mathbb{R} \cup \{-\infty, \infty\}$ with $a \leq b$, define the quadruply marked and possibly beaded quantum surfaces

$$\begin{aligned} \mathcal{S}_{a,b} &:= (\eta'([a, b]), h|_{\eta'([a, b])}, \eta'(a), \eta'(b), \eta'(x_{a,b}^L), \eta'(x_{a,b}^R)) \quad \text{and} \\ \tilde{\mathcal{S}}_{a,b} &:= (\tilde{\eta}'([a, b]), h|_{\tilde{\eta}'([a, b])}, \tilde{\eta}'(a), \tilde{\eta}'(b), \eta'(x_{a,b}^L), \eta'(x_{a,b}^R)) \end{aligned} \quad (3.2)$$

where here $x_{a,b}^L$ (resp. $x_{a,b}^R$) is the first time in $[a, b]$ at which the Brownian motion coordinate L (resp. R) attains its minimum value on $[a, b]$ or $x_{a,b}^L = x_{a,b}^R = \infty$ if $[a, b]$ is unbounded. By the peanosphere construction $\eta'(x_{a,b}^L)$ (resp. $\eta'(x_{a,b}^R)$) is the point in $\partial\eta'([a, b])$ where the left (resp. right) outer boundaries of $\eta'((-\infty, a])$ and

$\eta'([b, \infty))$ meet, and by condition 2 of Theorem 2.4, the same is true with $\tilde{\eta}'$ in place of η' . Note that in some degenerate cases (e.g., if either a or b is equal to ∞) some of the marked points of $\mathcal{S}_{a,b}$ are equal, so there are actually fewer than 4 marked points.

By condition 1 in Theorem 2.4, each $\tilde{\mathcal{S}}_{t,\infty}$ for $t \in \mathbb{R}$ is a $\frac{3\gamma}{2}$ -quantum wedge. By the proof of [DMS14, Lemma 9.2], $(\mathcal{S}_{-\infty,t}, \mathcal{S}_{t,\infty})$ for $t \in \mathbb{R}$ is a pair of independent $\frac{3\gamma}{2}$ -quantum wedges which are conformally welded according to quantum length along their boundaries to form \mathcal{C} .

For $a, b \in \mathbb{R} \cup \{-\infty, \infty\}$ with $a < b$, we define a curve $\eta_{a,b} : [a, b] \rightarrow \eta'([a, b])$ by skipping all of the bubbles filled in by η' during the time interval $[a, b]$. More precisely, if $t \in [a, b]$ is such that $\eta'(t)$ is contained in a bubble filled in by η' during the time interval $[a, b]$, we let $[\sigma_{a,b}(t), \tau_{a,b}(t)]$ be the time interval during which η' fills in the largest such bubble. Otherwise, we let $\sigma_{a,b}(t) = \tau_{a,b}(t) = t$. Then $\eta'(\sigma_{a,b}(t)) = \eta'(\tau_{a,b}(t))$, and we define $\eta_{a,b}(t)$ to be the common value of these two quantities. We note that the times $\sigma_{a,b}(t)$ and $\tau_{a,b}(t)$ can be recovered from $\eta_{a,b}$ by the formulas

$$\sigma_{a,b}(t) := \sup\{s \leq t : \eta_{a,b}(s) \neq \eta_{a,b}(t)\} \quad \text{and} \quad \tau_{a,b}(t) := \inf\{s \geq t : \eta_{a,b}(s) \neq \eta_{a,b}(t)\}. \quad (3.3)$$

In the special case when $a = -\infty$, the curve $\eta_{-\infty,b}$ is obtained from η' by cutting out the bubbles which η' disconnects from $\eta'(b)$. By translation invariance [DMS14, Lemma 9.3], $\eta_{-\infty,b}$ has the same law as $\eta_{-\infty,0}$, which is the chordal $\text{SLE}_{\kappa'}(\kappa' - 6)$ counterflow line from ∞ to 0 associated with η' . In general, $\eta_{a,b}$ is an $\text{SLE}_{\kappa'}$ -type curve contained in $\eta'([a, b])$.

Let $\mathcal{S}_{a,b}^0$ be the sub-surface of $\mathcal{S}_{a,b}$ parameterized by the set of bubbles cut out by $\eta_{a,b,\mathcal{S}_{a,b}}$, each marked by the point where they are cut off by $\eta_{a,b,\mathcal{S}_{a,b}}$. Equivalently, $\mathcal{S}_{a,b}^0$ is parameterized by the set of bubbles filled in by $\eta'_{\mathcal{S}_{a,b}}$ during the time interval $[a, b]$, each marked by the point where η' starts (equivalently finishes) filling it in. We view $\mathcal{S}_{a,b}^0$ as a quantum surface with (at most) 4 marked points, namely the ones it inherits from $\mathcal{S}_{a,b}$ (recall (3.2)). By Lemma 3.1 just below, the bubbles of $\mathcal{S}_{a,b}^0$ are the same as the set of singly-marked quantum surfaces $\mathcal{S}_{v_Z(s),s} = (\eta'([v_Z(s), s]), h|_{\eta'([v_Z(s), s])}, \eta'(t))$ where $[v_Z(s), s]$ ranges over all maximal $\frac{\pi}{2}$ -cone excursions for Z in $[a, b]$ (recall Section 2.3.4).

The surface $\mathcal{S}_{a,b}^0$ is the same as the equivalence class of $\mathcal{S}_{a,b}$ modulo homeomorphisms which are conformal on $\mathcal{S}_{a,b} \setminus \eta_{a,b,\mathcal{S}_{a,b}}$. Since we do not know that the $\text{SLE}_{\kappa'}$ -type curve $\eta_{a,b}$ is conformally removable, such a homeomorphism is not necessarily conformal on all of $\mathcal{S}_{a,b}$.

Define the curve $\tilde{\eta}_{a,b}$ and the surface $\tilde{\mathcal{S}}_{a,b}^0$ in the same manner as above but with $(\tilde{h}, \tilde{\eta}')$ in place of (h, η') . By condition 2 in Theorem 2.4, a.s. $\Phi \circ \tilde{\eta}_{a,b} = \eta_{a,b}$. Furthermore, the definitions of the times $\sigma_{a,b}(t)$ and $\tau_{a,b}(t)$ from (3.3) are unchanged if we replace (h, η') by $(\tilde{h}, \tilde{\eta}')$.

In the following lemma, we describe the times $\sigma_{a,b}(t)$ and $\tau_{a,b}(t)$ in terms of the $\frac{\pi}{2}$ -cone times s for Z and the associated cone entrance times $v_Z(s)$ (Definition 2.2). Recall also Definition 2.3 of a maximal $\frac{\pi}{2}$ -cone interval.

Lemma 3.1. *Suppose $a, b \in \mathbb{R} \cup \{-\infty, \infty\}$ with $a < b$. For $t \in [a, b]$, $[\sigma_{a,b}(t), \tau_{a,b}(t)]$ is the same as the maximal $\frac{\pi}{2}$ -cone interval for Z in $[a, b]$ containing t (Definition 2.3), if it exists, or the singleton $\{t\}$ otherwise. In particular, if $\tau_{a,b}(t) \neq t$ then $\tau_{a,b}(t)$ is a $\frac{\pi}{2}$ -cone time for Z and $v_Z(\tau_{a,b}(t)) = \sigma_{a,b}(t)$. Furthermore, the quantum area (resp. quantum boundary length) of the surface $\mathcal{S}_{\sigma_{a,b}(t), \tau_{a,b}(t)}$ is equal to $\tau_{a,b}(t) - \sigma_{a,b}(t)$ (resp. $|Z_{\tau_{a,b}(t)} - Z_{\sigma_{a,b}(t)}|$).*

Proof. This follows from the correspondence between bubbles filled in by η' and $\frac{\pi}{2}$ -cone excursions for Z ; c.f. Section 2.3.4. \square

We note that Lemma 3.1 and condition 2 in Theorem 2.4 imply that

$$(\tilde{\mathcal{S}}_{a,b}^0, \tilde{\eta}_{a,b}, \tilde{\mathcal{S}}_{a,b}^0) \in \tilde{\mathcal{F}}_{a,b} \quad (3.4)$$

where $\tilde{\mathcal{F}}_{a,b}$ is the σ -algebra defined just above Theorem 2.4.

The last object we introduce in this subsection is a function which encodes the areas and left/right quantum boundary lengths of the beads of $\mathcal{S}_{t,\infty}$ and $\tilde{\mathcal{S}}_{t,\infty}$. For $t \in \mathbb{R}$ and $s \geq t$, let

$$P_{t,\infty}(s) := \left(\bar{T}_t(s) - \underline{T}_t(s), L_{\underline{T}_t(s)} - L_{\bar{T}_t(s)}, R_{\underline{T}_t(s)} - R_{\bar{T}_t(s)} \right) \quad (3.5)$$

where $\bar{T}_t(s)$ (resp. $\underline{T}_t(s)$) is the first time after (resp. the last time before) s at which the Brownian motion coordinates L and R attain a simultaneous running infimum relative to time t . We also set $P_{t,\infty}(s) = (0, 0, 0)$ for $s \leq t$.

We remark that $P_{t,\infty}$ a.s. determines $(\bar{T}_t(s), L_{\bar{T}_t(s)}, R_{\bar{T}_t(s)})$ (by summing the distinct values taken by $P_{t,\infty}(s')$ for $s' \leq s$) since the set of times where L and R attain a simultaneous running infimum a.s. has Minkowski dimension $1 - \kappa'/8 < 1/2$ (c.f. Lemma 3.5 below) and Z is a.s. Hölder continuous of any exponent $< 1/2$, so the total variation of Z over this set is a.s. equal to 0.

The significance of the function $P_{t,\infty}$ is contained in the following lemma (c.f. Section 2.3.4).

Lemma 3.2. *Let $t \in \mathbb{R}$ and $s \geq t$. Almost surely, the coordinates of $P_{t,\infty}(s)$ are a.s. equal to the quantum area, left quantum boundary length, and right quantum boundary length of the bead of $\mathcal{S}_{t,\infty}$ containing $\eta'_{\mathcal{S}_{t,\infty}}(s)$, respectively. The same is true with $\tilde{\mathcal{S}}_{t,\infty}$ and $\tilde{\eta}'$ in place of $\mathcal{S}_{t,\infty}$ and η' .*

Proof. The beads of $\mathcal{S}_{t,\infty}$ are parameterized by the connected components of the interior of $\eta'([t, \infty))$, which are filled in order by $\eta'|_{[t, \infty)}$. The intervals of time during which $\eta'|_{[t, \infty)}$ is filling in one of these components are the same as the maximal time intervals in $[t, \infty)$ during which η' does not hit the left and right boundaries of $\eta'((-\infty, t])$ simultaneously. Since a time when $\eta'|_{[t, \infty)}$ hits its left (resp. right) boundary without forming a bubble is the same as a time when L (resp. R) attains a running infimum relative to time 0 and η' never forms a bubble at a time when L and R attain a simultaneous running infimum, we obtain the first statement of the lemma. The second statement follows from the first statement together with condition 2 in Theorem 2.4. \square

3.2 Measurability results for (h, η')

In this subsection and the next, we will consider the objects in Section 3.1 obtained from the γ -quantum cone/space-filling SLE $_{\kappa'}$ pair (h, η') ; we will generalize some of the results about these objects to the analogous objects defined in terms of $(\tilde{h}, \tilde{\eta}')$ in the later subsections. The focus of the present section is on measurability results for these objects, i.e., statements that some object a.s. determines another.

We will start by proving some measurability results for objects defined in terms of the correlated two-dimensional Brownian motion Z , then generalize to statements about (h, η') using [DMS14, Theorem 1.14]. Our first statement tells us that the restriction of Z to an interval $[a, b]$ is a.s. determined by the maximal $\frac{\pi}{2}$ -cone excursions of Z in $[a, b]$ (Definitions 2.2 and 2.3) plus a small amount of additional information. For the statement of the lemma, we recall the characterization of the times $\sigma_{a,b}(t)$ and $\tau_{a,b}(t)$ from Lemma 3.1.

Lemma 3.3. *Let $a, b \in \mathbb{R} \cup \{\infty\}$ with $a < b$. Let $\mathcal{M}_{a,b}$ be the set of times in $[a, b]$ at which the coordinates of Z attain a simultaneous running infimum relative to time a and let*

$$\mathcal{H}_{a,b} := \sigma(\sigma_{a,b}(t), \tau_{a,b}(t), (Z - Z_{\tau_{a,b}(t)})|_{[\sigma_{a,b}(t), \tau_{a,b}(t)]} : t \in [a, b]) \vee \sigma(\mathcal{M}_{a,b}).$$

Then $(Z - Z_a)|_{[a,b]}$ is $\mathcal{H}_{a,b}$ -measurable.

The set

$$[a, b] \setminus \bigcup_{t \in [a, b]} (\sigma_{a,b}(t), \tau_{a,b}(t)) \tag{3.6}$$

a.s. has zero Lebesgue measure (in fact, we will see in Proposition 4.7 below that its Minkowski dimension is at most $\kappa'/8$). This fact does not imply Lemma 3.3 since there can be “local time” fluctuations of Z on this small set, so it is not obvious that there is a unique way to concatenate the excursions $(Z - Z_{\tau_{a,b}(t)})|_{[\sigma_{a,b}(t), \tau_{a,b}(t)]}$ to recover $(Z - Z_a)|_{[a,b]}$.

We will deduce Lemma 3.3 from the following stronger statement for the case when $a = -\infty$. The reason why we can get a stronger statement in this case is that the set (3.6) is regenerative when $a = -\infty$ (but not for any other values of a).

Lemma 3.4. *Let $b \in \mathbb{R}$ and let for $r \leq b$, let*

$$\mathcal{H}'_{-\infty,b}(r) := \sigma((Z - Z_{\tau_{-\infty,b}(t)})|_{[r,b]}).$$

Then $(Z - Z_b)|_{[r,b]}$ is $\mathcal{H}'_{-\infty,b}(r)$ -measurable.

Proof. By translation invariance we can assume without loss of generality that $b = 0$, so that $Z_b = 0$. Condition on $\mathcal{H}'_{-\infty,0}(r)$ and sample Z^1 and Z^2 from the regular conditional law of Z given $\mathcal{H}'_{-\infty,0}(r)$ in such a way that they are conditionally independent given $\mathcal{H}'_{-\infty,0}(r)$. Then $Z^1 \stackrel{d}{=} Z^2 \stackrel{d}{=} Z$; for each $t \in [r, 0]$, $\tau_{-\infty,0}(t)$ is the right endpoint of the maximal $\frac{\pi}{2}$ -cone interval in $(-\infty, 0]$ containing t for each of Z^1 and Z^2 ; and for each $t \in [r, 0]$, $Z_t^1 - Z_{\tau_{-\infty,0}(t)}^1 = Z_t^2 - Z_{\tau_{-\infty,0}(t)}^2$. We must show that $Z^1|_{[r,0]} = Z^2|_{[r,0]}$ a.s.

To see this, we consider the discontinuous process

$$W_t := Z_t - Z_{\tau_{-\infty,0}(t)}, \quad \forall t \leq 0.$$

Since $\tau_{-\infty,0}(t)$ is the smallest $s \geq t$ for which $W_s = 0$, it follows that $\sigma(W|_{[t,0]}) = \mathcal{H}'_{-\infty,0}(t)$. Since each $\tau_{-\infty,0}(t)$ is determined by $Z|_{[t,0]}$,

$$W|_{[t,0]} \in \sigma(Z^1|_{[t,0]}) \cap \sigma(Z^2|_{[t,0]}).$$

For $t \leq 0$, the regular conditional law of $W|_{(-\infty,t]}$ given $Z^1|_{[t,0]}$ can be described as follows. The process $W|_{(-\infty,t]}$ evolves as a Brownian motion with variances and covariances as in (1.2) started from W_t at time t and run backward until the first time it exits the cone $W_{\tau_{-\infty,0}(t)} + [0, \infty)^2$ (this time is $\sigma_{-\infty,0}(t)$). Conditional on $W|_{[\sigma_{-\infty,0}(t),0]}$, the rest of the process, $W|_{(-\infty,\sigma_{-\infty,0}(t))}$, has the same law as W . This conditional law depends only on $W|_{[t,0]}$, so $W|_{(-\infty,t]}$ is conditionally independent from $Z^1|_{[t,0]}$ given $W|_{[t,0]}$.

By our choice of coupling, for each $t \in [r, 0]$ the regular conditional law of Z^2 given $Z^1|_{[t,0]}$ and W depends only on W . Since the conditional law of $W|_{(-\infty,t]}$ given $Z^1|_{[t,0]}$ depends only on $W|_{[t,0]}$, we infer that the conditional law of Z^2 given $Z^1|_{[t,0]}$ depends only on $W|_{[t,0]}$. Since $W|_{[t,0]}$ is determined by $Z^2|_{[t,0]}$, the regular conditional law of $(Z^2 - Z_t^2)|_{(-\infty,t]}$ given $Z^1|_{[t,0]}$ and $Z^2|_{[t,0]}$ is the same as its regular conditional law given only $Z^2|_{[t,0]}$, which is the same as the law of Z , run backward starting from 0.

In particular, if we set

$$\mathcal{G}_t := \sigma(Z^1|_{[t,0]}, Z^2|_{[t,0]})$$

then for $s \leq t$ it holds that $\mathbb{E}[Z_s^2 | \mathcal{G}_t] = Z_t^2$. By symmetry, $\mathbb{E}[Z_s^1 | \mathcal{G}_t] = Z_t^1$. Therefore, $Z^1 - Z^2$ is a backward continuous $\{\mathcal{G}_t\}_{t \leq 0}$ -martingale. By our choice of coupling, $Z^1 - Z^2$ is constant on the intersection with $[r, 0]$ of every maximal $\frac{\pi}{2}$ -cone interval for Z^1 (equivalently, for Z^2) in $(-\infty, 0]$.

The set $\mathcal{R} := (-\infty, 0] \setminus \bigcup_{t \leq 0} (\sigma_{-\infty,0}(t), \tau_{-\infty,0}(t))$ is the set of so-called ancestor free times for Z , run backward from time 0 (see [DMS14, Section 12.1]). The set \mathcal{R} is easily seen to be regenerative and scale-invariant, so has the law of the range of a stable subordinator run backward from time 0. The Hausdorff dimension of \mathcal{R} is a.s. equal to $\kappa'/8$ (see, e.g., [GHM15, Example 2.3]), so the index of this subordinator is $\kappa'/8$. In particular, the Minkowski dimension of the set $\mathcal{R} \cap [r, 0]$ is a.s. equal to $\kappa'/8$. Since $Z^1 - Z^2$ is constant on $[r, 0] \setminus \mathcal{R}$ and is a.s. Hölder continuous of any exponent less than $1/2$, and $\kappa'/8 < 1$ we infer that the quadratic variation of $Z^1 - Z^2$ on $[r, 0]$ is a.s. equal to 0. Therefore, $(Z^1 - Z^2)|_{[r,0]} = 0$ a.s. \square

Before we can deduce Lemma 3.3 from Lemma 3.4, we first need the following fact about the simultaneous running infima of L and R which will tell us that the set $\mathcal{M}_{a,b}$ of Lemma 3.3 is in some sense negligible.

Lemma 3.5. *Let $\mathcal{M}_{0,\infty}$ be the set of times $s \geq 0$ at which L and R attain a simultaneous running infimum relative to time 0. Then $\mathcal{M}_{0,\infty}$ has the law of the range of a $1 - \kappa'/8$ -stable subordinator.*

Proof. It is clear from the Markov property of Brownian motion that $\mathcal{M}_{0,\infty}$ is a scale-invariant regenerative set, so is either empty or has the law of the range of a stable subordinator. A time at which L and R attain a simultaneous running infimum relative to time 0 is the same as a $\frac{\pi}{2}$ -cone time for Z whose corresponding $\frac{\pi}{2}$ -cone interval $[v_Z(s), s]$ contains 0. Hence it follows from [Eva85, Theorem 1] (c.f. the proofs of [DMS14, Lemmas 9.4 and 9.5]) that the Hausdorff dimension of $\mathcal{M}_{0,\infty}$ is a.s. equal to $1 - \kappa'/8$ if $\kappa' \in (4, 8)$. The statement of the lemma follows. \square

Proof of Lemma 3.3. For $r_1, r_2 \in [a, b]$ with $r_1 < r_2$, let

$$E(r_1, r_2) := \{[r_1, r_2] \cap \mathcal{M}_{a,b} = \emptyset\},$$

and note that $E(r_1, r_2) \in \mathcal{H}_{a,b}$ by definition.

If $t \in [r_1, r_2]$ and $\sigma_{-\infty,r_2}(t) < a$, then $\tau_{-\infty,r_2}(t) \in \mathcal{M}_{a,b} \cap [t, r_2]$. Therefore, on the event $E(r_1, r_2)$ we have $\sigma_{-\infty,r_2}(t) \geq a$ for each $t \in [r_1, r_2]$, whence the $\frac{\pi}{2}$ -cone interval $[\sigma_{-\infty,r_2}(t), \tau_{-\infty,r_2}(t)]$ is entirely contained in

the maximal $\frac{\pi}{2}$ -cone interval $[\sigma_{a,b}(t), \tau_{a,b}(t)]$ in $[a, b]$. The times $\sigma_{-\infty, r_2}(t)$ and $\tau_{-\infty, r_2}(t)$ are determined by $(Z - Z_{\tau_{a,b}(t)})|_{[\sigma_{a,b}(t), \tau_{a,b}(t)]}$ on this event since $[\sigma_{-\infty, r_2}(t), \tau_{-\infty, r_2}(t)]$ must be the maximal $\frac{\pi}{2}$ -cone interval for Z in $[\sigma_{a,b}(t), r_2]$ which contains t .

The preceding paragraph implies that on $E(r_1, r_2)$, each of the Brownian cone excursions $(Z - Z_{\tau_{-\infty, r_2}(t)})$ in the interval of time $[\sigma_{-\infty, r_2}(t), \tau_{-\infty, r_2}(t)]$ for $t \in [r_1, r_2]$ is a.s. determined by $\mathcal{H}_{a,b}$. By Lemma 3.4, on $E(r_1, r_2)$ the σ -algebra $\mathcal{H}_{a,b}$ a.s. determines $(Z - Z_{r_2})|_{[r_1, r_2]}$. Exhausting over all rational values of $r_1, r_2 \in [a, b]$, we find that $\mathcal{H}_{a,b}$ a.s. determines $Z_t - Z_{\underline{T}_a(t)}$ for each $t \in [a, b]$, where $\underline{T}_a(t)$ is the largest $s \leq t$ such that $s \in \mathcal{M}_{a,b}$ (as in (3.5)).

The set $\mathcal{M}_{a,b}$ has the law of range of a $1 - \kappa'/8$ -stable subordinator, so a.s. has Minkowski dimension $1 - \kappa'/8 < 1/2$. Since Z is a.s. Hölder continuous for any exponent smaller than $1/2$, it follows that the total variation of Z over $\mathcal{M}_{a,b}$ is a.s. equal to zero. Therefore, $(Z - Z_a)|_{[a,b]}$ is a.s. determined by $\{Z_t - Z_{\underline{T}_a(t)} : t \in [a, b]\}$, and hence by $\mathcal{H}_{a,b}$. \square

Now we turn our attention to measurability statements for objects defined in terms of the pair (h, η') . For this purpose we first need the following statement about the law of $\eta'|_{[a,b]}$.

Lemma 3.6. *Let $a < b$. The conditional law of the curve $\eta'|_{[a,b]}$ given $\eta'|_{\mathbb{R} \setminus [a,b]}$ and h is that of a concatenation of independent chordal space-filling $\text{SLE}_{\kappa'}$ curves in the interior connected components of $\eta'([a, b])$, whose laws are described as follows.*

- For each interior connected component U of $\eta'([a, b])$ whose boundary is part of $\partial\eta'((-\infty, a])$, the conditional law of the segment of η' contained in \bar{U} is that of a chordal space-filling $\text{SLE}_{\kappa'}$ in U between the points where η' starts and finishes filling in U .
- For the interior connected component U_* of $\eta'([a, b])$ whose boundary contains non-trivial arcs of each of $\partial\eta'((-\infty, a])$ and $\partial\eta'([b, \infty))$, the conditional law of the segment of η' contained in \bar{U}_* is that of a chordal space-filling $\text{SLE}_{\kappa'}(\kappa'/2 - 4; \kappa'/2 - 4)$ in U_* between the points where η' starts and finishes filling in U , with force points located at the two points of ∂U_* where $\partial\eta'((-\infty, a])$ and $\partial\eta'([b, \infty))$ meet.
- For each interior connected component U of $\eta'([a, b])$ whose boundary is part of $\partial\eta'([b, \infty))$, the conditional law of the segment of η' contained in \bar{U} is that of a chordal space-filling $\text{SLE}_{\kappa'}(\kappa'/2 - 4; \kappa'/2 - 4)$ in U between the points where η' starts and finishes filling in U , with force points located immediately to the left and right of the starting point.

Proof. Let h^{IG} be the whole-plane GFF, viewed modulo a global additive multiple of $2\pi\chi$, used to construct η' as in Section 2.3.2, where here χ is as in (2.8) (IG stands for “imaginary geometry”). Recall that the outer boundary of η' at the first time it hits and rational $z \in \mathbb{C}$ is the union of the flow lines of h^{IG} started from z with angles $\pm\frac{\pi}{2}$.

By translation invariance of η' [DMS14, Lemma 9.3] and [DMS14, Footnote 9], the conditional law of $\eta'|_{[a, \infty)}$ given h and $\eta'|_{(-\infty, a]}$ is that of a concatenation of chordal space-filling $\text{SLE}_{\kappa'}$ curves in the interior connected components of $\eta'([a, \infty))$. The time b is a reverse stopping time for the conditional law of $\eta'|_{[a, \infty)}$ given h and $\eta'|_{(-\infty, a]}$ (b is the largest $t \geq a$ such that the set $\mathbb{C} \setminus (\eta'((-\infty, a]) \cup \eta'([t, \infty)))$ has quantum mass $b - a$). Hence the conditional law of the field $h^{\text{IG}}|_{\eta'([a, b])}$ given $\eta'|_{\mathbb{R} \setminus [a, b]}$ and h is that of an independent GFF with Dirichlet boundary conditions in each of the interior connected components of $\eta'([a, b])$ specified by the boundary data for h along the outer boundaries of $\eta'((-\infty, a])$ and $\eta'([b, \infty))$. In particular, using the description of flow line boundary data in [MS13, Theorem 1.1], we find that with $\lambda' := \pi/\sqrt{\kappa'}$ and χ as above,

- The boundary data for $h^{\text{IG}}|_{\eta'([a, b])}$ along the left (resp. right) outer boundary of $\eta'((-\infty, a])$ is given by $\lambda' + \chi \arg \phi'$ (resp. $-\lambda' + \chi \arg \phi'$), where ϕ denotes a conformal map which takes a given interior connected component of $\eta'([a, b])$ to \mathbb{H} in such a way that its intersection with the left (resp. right) outer boundary of $\eta'((-\infty, a])$ is sent to $(-\infty, 0]$ (resp. $[0, \infty)$).
- The boundary data for $h^{\text{IG}}|_{\eta'([a, b])}$ along the left (resp. right) outer boundary of $\eta'([b, \infty))$ is given by $-\lambda'(\kappa'/2 - 3) + \chi \arg \phi'$ (resp. $\lambda'(\kappa'/2 - 3) + \chi \arg \phi'$), with ϕ as above.

From the construction of space-filling $\text{SLE}_{\kappa'}$ in [MS13, Section 1.2.3], we see that the segment of η' contained in the closure of each interior connected component of $\eta'([a, b])$ is equal to the space-filling $\text{SLE}_{\kappa'}$ counterflow line of $h^{\text{IG}}|_U$. The above description of the boundary data for $h^{\text{IG}}|_U$ implies that the conditional law of η' must be as in the statement of the lemma. \square

We next prove a lemma which implies in particular that the quantum surface decorated by an $\text{SLE}_{\kappa'}$ -type curve $(\mathcal{S}_{a,b}, \eta_{a,b}, \mathcal{S}_{a,b})$ is a.s. determined by the curve-decorated quantum surface $(\mathcal{S}_{a,\tau}^0, \eta_{a,b}, \mathcal{S}_{a,\tau}^0)$ from Section 3.1, which we recall only encodes the conformal structure and the topology of the bubbles cut out by $\eta_{a,b}, \mathcal{S}_{a,b}$ (rather than the full conformal structure of $\mathcal{S}_{a,b}$). One should think of this result as an analog of the measurability statements in [DMS14, Theorems 1.17 and 1.18] for the pair $(\mathcal{S}_{a,b}, \eta_{a,b}, \mathcal{S}_{a,b})$.

Lemma 3.7. *Let $a, t, b \in \mathbb{R}$ with $a \leq t \leq b$ and let $\tau = \tau_{a,b}(t)$ be as in (3.3). The following measurability statements hold.*

1. $(\mathcal{S}_{a,\tau}, \eta'_{\mathcal{S}_{a,\tau}})$ is a.s. determined by $(\mathcal{S}_{a,\tau}^0, \eta'_{\mathcal{S}_{a,\tau}})$.
2. $(\mathcal{S}_{a,\tau}, \eta_{a,b}, \mathcal{S}_{a,\tau})$ is a.s. determined by $(\mathcal{S}_{a,\tau}^0, \eta_{a,b}, \mathcal{S}_{a,\tau}^0)$.

Proof. We first prove assertion 2. It is clear that $\mathcal{S}_{a,\tau}^0$ a.s. determines τ , since $\tau - a$ is the total quantum mass of $\mathcal{S}_{a,\tau}^0$. Recalling Lemma 3.1, we find that $(\mathcal{S}_{a,\tau}^0, \eta'_{\mathcal{S}_{a,\tau}})$ a.s. determines the Brownian cone excursions $(Z - Z_t)|_{[v_Z(t), t]}$ for each maximal $\frac{\pi}{2}$ -cone time t for Z in $[a, b]$. Furthermore, $\mathcal{S}_{a,\tau}^0$ a.s. determines the quantum mass of each bead of $\mathcal{S}_{a,\tau}$, which together with the last two marked points of $\mathcal{S}_{a,b}^0$ from (3.2) determines the times in $[a, b]$ at which the two coordinates of Z attain a simultaneous running infimum (Lemma 3.2). Hence Lemma 3.3 implies that for each $r \in [a, b]$, $(\mathcal{S}_{a,\tau}^0, \eta'_{\mathcal{S}_{a,\tau}})$ a.s. determines $(Z - Z_a)|_{[a,r]}$ on the event $\{r \leq \tau\}$. Exhausting over all rational $r \in [a, b]$ shows that $(\mathcal{S}_{a,\tau}^0, \eta'_{\mathcal{S}_{a,\tau}})$ a.s. determines $(Z - Z_a)|_{[a,\tau]}$. It follows from the results of [DMS14] (see [GHS16b, Lemma 3.12] for a careful explanation) that for each $r \in [a, b]$, $(Z - Z_a)|_{[a,\tau]}$ a.s. determines $(\mathcal{S}_{a,r}, \eta'_{\mathcal{S}_{a,r}})$ on the event $\{r \leq \tau\}$. Again exhausting over all rational $r \in [a, b]$, we obtain assertion 1.

Now we deduce assertion 2 from assertion 1. Although τ is not a stopping time for Z , it is a stopping time for the conditional law of the curve $\eta_{a,b}, \mathcal{S}_{a,b}$ given $\mathcal{S}_{a,b}$. Hence we can use Lemma 3.6 and the Markov property of chordal space-filling $\text{SLE}_{\kappa'}(\rho^L; \rho^R)$ to obtain that if we condition on $(\mathcal{S}_{a,\tau}, \eta_{a,b}, \mathcal{S}_{a,\tau})$ then the regular conditional law of $\eta'_{\mathcal{S}_{a,\tau}^0}$ is that of an independent space-filling $\text{SLE}_{\kappa'}(\kappa' - 6)$ loop in each of the bubbles of $\mathcal{S}_{a,\tau}^0$, based at the marked point of the bubble. This conditional law depends only on $(\mathcal{S}_{a,\tau}^0, \eta_{a,b}, \mathcal{S}_{a,\tau}^0)$, so we get the same conditional law if we condition only on $(\mathcal{S}_{a,\tau}^0, \eta_{a,b}, \mathcal{S}_{a,\tau}^0)$. Hence $(\mathcal{S}_{a,\tau}^0, \eta'_{\mathcal{S}_{a,\tau}^0})$ is conditionally independent from $(\mathcal{S}_{a,\tau}, \eta_{a,b}, \mathcal{S}_{a,\tau})$ given $(\mathcal{S}_{a,\tau}^0, \eta_{a,b}, \mathcal{S}_{a,\tau}^0)$. By assertion 1, $(\mathcal{S}_{a,\tau}^0, \eta'_{\mathcal{S}_{a,\tau}^0})$ a.s. determines $(\mathcal{S}_{a,\tau}, \eta_{a,b}, \mathcal{S}_{a,\tau})$, so in fact $(\mathcal{S}_{a,\tau}^0, \eta_{a,b}, \mathcal{S}_{a,\tau}^0)$ a.s. determines $(\mathcal{S}_{a,\tau}, \eta_{a,b}, \mathcal{S}_{a,\tau})$, which is assertion 2. \square

From Lemma 3.7, we easily deduce a measurability statement for the whole γ -quantum cone \mathcal{C} from (3.1).

Lemma 3.8. *Let $a, t, b \in \mathbb{R}$ with $a \leq t \leq b$ and let $\tau = \tau_{a,b}(t)$ be as in (3.3). The γ -quantum cone \mathcal{C} , the future $2 - \gamma^2/2$ -quantum wedge $\mathcal{S}_{a,\infty}$, and the curve $\eta_{a,b}, \mathcal{C}|_{[a,t]}$ are a.s. determined by the 4-tuple $(\mathcal{S}_{-\infty,a}, \mathcal{S}_{a,\tau}^0, \eta_{a,b}, \mathcal{S}_{a,\tau}^0, \mathcal{S}_{\tau,\infty})$.*

Proof. By Lemma 3.7, the curve-decorated quantum surface $(\mathcal{S}_{a,\tau}^0, \eta_{a,b}, \mathcal{S}_{a,\tau}^0)$ a.s. determines $(\mathcal{S}_{a,\tau}, \eta_{a,b}, \mathcal{S}_{a,\tau})$. By the peanosphere construction, the surface $\mathcal{S}_{a,\infty}$ is obtained by conformally welding together $\mathcal{S}_{a,\tau}$ and $\mathcal{S}_{\tau,\infty}$ along their boundaries and the surface \mathcal{C} is obtained by conformally welding together $\mathcal{S}_{-\infty,a}$ and $\mathcal{S}_{a,\infty}$ together along their boundaries. The boundary of each of the space-filling $\text{SLE}_{\kappa'}$ segments $\eta'((-\infty, a])$, $\eta'([a, \tau])$, and $\eta'([\tau, \infty))$ consists of a finite union of segments of non-crossing $\text{SLE}_{\kappa'}$ -type curves for $\kappa = 16/\kappa' \in (0, 4)$, so is a.s. conformally removable by [DMS14, Proposition 1.8]. Hence there is a.s. only one way to perform the above conformal welding operations, and we obtain the statement of the lemma. \square

3.3 Bubbles cut out by $\eta_{a,b}$ are quantum disks

Suppose we are in the setting of Section 3.1, and recall in particular the chordal $\text{SLE}_{\kappa'}$ -type curves $\eta_{a,b} : [a, b] \rightarrow \eta'([a, b])$, the associated maximal $\frac{\pi}{2}$ -cone intervals $[\sigma_{a,b}(t), \tau_{a,b}(t)]$ from (3.3), and the surface $\mathcal{S}_{a,b}^0$ parameterized by the bubbles cut out by $\eta_{a,b}$. The goal of this subsection is to prove that the bubbles of $\mathcal{S}_{a,b}^0$ are conditionally independent quantum disks if we condition on the values of Z outside of the time intervals corresponding to these bubbles. In particular, we will prove the following proposition.

Proposition 3.9. *Let $a, b \in \mathbb{R} \cup \{-\infty, \infty\}$ with $a < b$. Almost surely, the conditional law of the bubbles of the quantum surface $\mathcal{S}_{a,b}^0$ given $Z|_{(-\infty,a]}$, $Z|_{[b,\infty)}$, and $\{\tau_{a,b}(t) : t \in [a,b]\}$ is that of a collection of independent singly marked quantum disks with given areas and boundary lengths.*

Recall that the bubbles of $\mathcal{S}_{a,b}^0$ are the same as the quantum surfaces $\mathcal{S}_{\sigma_{a,b}(t), \tau_{a,b}(t)}$ for $t \in (a,b)$ with $\tau_{a,b}(t) \neq t$. The main difficulty in the proof of Proposition 3.9 lies in showing that each of these bubbles is a quantum disk conditional on its quantum area and boundary length.

Throughout most of this subsection, we fix $a, b \in \mathbb{R} \cup \{-\infty, \infty\}$ with $a < b$ and $t \in (a,b)$ and to lighten notation we write

$$\tau = \tau_{a,b}(t) \quad \text{and} \quad \sigma = \sigma_{a,b}(t).$$

We will only allow t to vary at the very end, when we argue that the bubbles are conditionally independent. The idea of the proof is to relate the bubble $\mathcal{S}_{\tau,\sigma}$ to the bubbles cut out by the $\text{SLE}_{\kappa'}(\kappa' - 6)$ curves $\eta_{-\infty,r}$ for $r \in \mathbb{R}$, which we know are quantum disks by [DMS14, Theorem 1.18].

Throughout the proof, we use the following notation. For $r \geq t$, let $\sigma^r = \sigma_{-\infty,r}(t)$ and $\tau^r = \tau_{-\infty,r}(t)$ be as in (3.3) with $a = -\infty$ and $b = r$. Equivalently, by Lemma 3.1, $[\sigma^r, \tau^r]$ is the maximal $\frac{\pi}{2}$ -cone interval for Z in $(-\infty, r]$ which contains t . Define the σ -algebra

$$\mathcal{F}^r := \sigma(\sigma^r, \tau^r, Z|_{\mathbb{R} \setminus [\sigma^r, \tau^r]}). \quad (3.7)$$

Lemma 3.10. *Suppose we are in the setting described just above. For $r' \geq r \geq t$, we have $\mathcal{F}^{r'} \subset \mathcal{F}^r$, so $\{\mathcal{F}^r\}_{r \geq t}$ is a decreasing filtration. Furthermore, if we let*

$$\rho = \sup\{r \geq t : [\sigma^r, \tau^r] = [\sigma, \tau]\} \quad (3.8)$$

then ρ is a reverse stopping time for $\{\mathcal{F}^r\}_{r \geq t}$ and a.s. $\rho > t$. Furthermore, there a.s. exists $\delta > 0$ such that $[\sigma^r, \tau^r] = [\sigma, \tau]$ for each $r \in [\rho - \delta, \rho]$.

Proof. The σ -algebra \mathcal{F}^r determines σ^r , τ^r , and Z_s for each $s \in \mathbb{R} \setminus [\sigma^r, \tau^r]$. This information determines each $\frac{\pi}{2}$ -cone interval for Z which is not contained in $[\sigma^r, \tau^r]$, so in particular determines the endpoints of the maximal $\frac{\pi}{2}$ -cone interval for Z in I which contains $[\sigma^r, \tau^r]$ for each open interval (possibly unbounded) interval in \mathbb{R} which contains $[\sigma^r, \tau^r]$.

If $r' \geq r$, then $[\sigma^r, \tau^r]$ is a $\frac{\pi}{2}$ -cone interval in $(-\infty, r']$ so is contained in $[\sigma^{r'}, \tau^{r'}]$. Hence the preceding paragraph implies that $\sigma^{r'}, \tau^{r'} \in \mathcal{F}^r$. It is clear that $\mathcal{F}^{r'}$ is determined by $[\sigma^{r'}, \tau^{r'}]$ and Z_s for $s \in \mathbb{R} \setminus [\sigma^{r'}, \tau^{r'}]$, whence $\mathcal{F}^{r'} \subset \mathcal{F}^r$.

We next argue that $\rho > t$ a.s., i.e. there a.s. exists some $r > t$ for which $[\sigma^r, \tau^r] = [\sigma, \tau]$. By symmetry, we can assume without loss of generality that τ is a left $\frac{\pi}{2}$ -cone time for Z , i.e. $R_\sigma = R_\tau$. Almost surely, σ is not a local minimum for R and a.s. $\sigma > a$, so there a.s. exists $\epsilon > 0$ and $s_* \in (a, \sigma)$ such that $R_{s_*} \leq R_\tau - \epsilon$. By continuity, there a.s. exists $\delta > 0$ such that $R_s \geq R_\tau - \epsilon$ for each $s \in [\tau, \tau + \delta]$. If s is a $\frac{\pi}{2}$ -cone time for Z belonging to $(\tau, \tau + \delta]$, then $v_Z(s) \geq s_* \geq a$ so by maximality of $[\sigma, \tau]$, we must have $v_Z(s) \geq \tau$.

If $r \in (\tau, \tau + \delta]$, then $[\sigma, \tau]$ is a $\frac{\pi}{2}$ -cone interval for Z in $(-\infty, r]$ so by maximality of $[\sigma^r, \tau^r]$, we have $\tau^r \in [\tau, \tau + \delta]$ and $\sigma^r \leq \sigma$. By the preceding paragraph, $[\sigma^r, \tau^r] = [\sigma, \tau]$ for each such r . Hence $\rho \geq \tau + \delta$ and $[\sigma^r, \tau^r] = [\sigma, \tau]$ for each $r \in [\rho - \delta, \rho]$.

Since $\rho > t$ a.s. and the intervals $[\sigma^r, \tau^r]$ are decreasing in r , if $r \geq t$ then a.s. $\rho \geq r$ if and only if $[\sigma^r, \tau^r] \subset [a, b]$. This latter event is \mathcal{F}^r -measurable, so $\{\rho \geq r\} \in \mathcal{F}^r$. \square

Our next lemma tells us the conditional law of the bubble $\mathcal{S}_{\sigma^r, \tau^r}$ given \mathcal{F}^r .

Lemma 3.11. *If $r \geq t$ and we condition on the σ -algebra \mathcal{F}^r of (3.7), then the conditional law of the quantum surface $\mathcal{S}_{\sigma^r, \tau^r}$ parameterized by $\eta'([\sigma^r, \tau^r])$ is that of a singly marked quantum disk with area $\sigma^r - \tau^r$ and boundary length $|Z_{\tau^r} - Z_{\sigma^r}|$.*

Proof. Recall that the curve $\eta_{-\infty,r}$ is a whole-plane $\text{SLE}_{\kappa'}(\kappa' - 6)$ from ∞ to $\eta'(r)$. Furthermore, by translation invariance of the law of the pair (h, η') [DMS14, Lemma 9.3], $(\mathbb{C}, h, \eta'(r), \infty)$ is a γ -quantum cone independent from $\eta_{-\infty,r}$ (viewed as a curve modulo monotone re-parameterization).

Let \mathcal{H}_1^r be the σ -algebra generated by the curve-decorated quantum surface $(\mathcal{S}_{r,\infty}, \eta'_{\mathcal{S}_{r,\infty}})$ and the quantum areas and the quantum boundary lengths of all of the bubbles of the quantum surfaces $\mathcal{S}_{-\infty,r}^0$ in the order in

which they are cut out by $\eta_{-\infty,r}$. Note that $\mathcal{S}_{-\infty,r}^0$, and $\eta_{-\infty,r,\mathcal{S}_{-\infty,r}^0}$ are determined by $(\mathcal{S}_{-\infty,r}, \eta'_{\mathcal{S}_{-\infty,r}})$ so are independent from $(\mathcal{S}_{r,\infty}, \eta'_{\mathcal{S}_{r,\infty}})$.

By [DMS14, Theorem 1.18] and the Markov property of whole-plane space-filling $\text{SLE}_{\kappa'}$, if we condition on \mathcal{H}_1^r then the conditional law of the curve-decorated quantum surface $(\mathcal{S}_{-\infty,r}^0, \eta'_{\mathcal{S}_{-\infty,r}^0})$ is that of a collection of independent singly marked quantum disks with given areas and boundary lengths, each decorated by a space-filling $\text{SLE}_{\kappa'}(\kappa' - 6)$ loop based at the marked point.

We can determine from \mathcal{H}_1^r which of the bubbles of $\mathcal{S}_{-\infty,r}^0$ is $\mathcal{S}_{\sigma^r,\tau^r}$ as follows. If we read off the quantum areas of the bubbles of $\mathcal{S}_{-\infty,r}^0$ in reverse chronological order, then $\mathcal{S}_{\sigma^r,\tau^r}$ is the first bubble which we encounter with the property that the sum of the quantum areas of the previous bubbles is at least t .

Consequently, the conditional law of $\mathcal{S}_{\sigma^r,\tau^r}$ given \mathcal{H}_1^r is that of a singly marked quantum disk with given area and boundary length. Furthermore, by the above independence statement, we get the same conditional law for $\mathcal{S}_{\sigma^r,\tau^r}$ if we further condition on the σ -algebra \mathcal{H}_2^r generated by \mathcal{H}_1^r and the curve-decorated quantum surfaces $(\mathcal{D}, \eta'_{\mathcal{D}})$ for each bubble \mathcal{D} of $\mathcal{S}_{-\infty,r}^0$ other than $\mathcal{S}_{\sigma^r,\tau^r}$.

By the relationship between $\frac{\pi}{2}$ -cone excursions for Z and bubbles filled in by η' , we have $\mu_h(\mathcal{S}_{\sigma^r,\tau^r}) = \sigma^r - \tau^r$ and $\nu_h(\partial\mathcal{S}_{\sigma^r,\tau^r}) = |Z_{\tau^r} - Z_{\sigma^r}|$ (note that one of the two coordinates of $Z_{\tau^r} - Z_{\sigma^r}$ is zero and the other is negative). Hence it remains to show that a.s. $\mathcal{F}^r \subset \mathcal{H}_2^r$. The σ -algebra \mathcal{H}_2^r a.s. determines $(Z - Z_r)|_{[r,\infty)}$, $\sigma^r - \tau^r$, $Z_{\tau^r} - Z_{\sigma^r}$, and $(Z - Z_{v_Z(s)})|_{[v_Z(s),s]}$ for each maximal $\frac{\pi}{2}$ -cone time s for Z in $(-\infty, r]$ other than $[\sigma^r, \tau^r]$.

As explained in [DMS14, Proposition 12.3], the complement of the union of all of the maximal $\frac{\pi}{2}$ -cone excursions for Z in $(-\infty, r]$ (the so-called ancestor free times for Z , run backward from time r) has the law of the range of $\kappa'/8$ -stable subordinator, and if we pre-compose Z with this stable subordinator we obtain a pair of independent $\kappa'/4$ -stable processes. These $\kappa'/4$ -stable processes are a.s. determined by their jumps, which in turn are a.s. determined by the quantities $Z_s - Z_{v_Z(s)}$ as s ranges over all maximal $\frac{\pi}{2}$ -cone times for Z in $(-\infty, r]$. From this, we deduce that the maximal $\frac{\pi}{2}$ -cone excursions $(Z - Z_{v_Z(s)})|_{[v_Z(s),s]}$ for Z in $(-\infty, r]$ other than $[\sigma^r, \tau^r]$ a.s. determine the values of $Z - Z_r$ on $(-\infty, r] \setminus [\sigma^r, \tau^r]$. Hence $\mathcal{F}^r \subset \mathcal{H}_2^r$. \square

Proof of Proposition 3.9. We first consider the single bubble $\mathcal{S}_{\sigma,\tau}$. Let ρ be as in (3.8) and for $n \in \mathbb{N}$, let $\rho_n = 2^{-n}\lfloor 2^n \rho \rfloor$ be the largest integer multiple of 2^{-n} which is smaller than ρ . By Lemma 3.10, each ρ_n is a reverse stopping time for the reverse filtration $\{\mathcal{F}^r\}_{r \geq t}$ and a.s. ρ_n increases to ρ . By Lemma 3.11, for each $n \in \mathbb{N}$ the conditional law of $\mathcal{S}_{\sigma^{\rho_n}, \tau^{\rho_n}}$ given \mathcal{F}^{ρ_n} is that of a singly marked quantum disk with area $\tau^{\rho_n} - \sigma^{\rho_n}$ and boundary length $|Z_{\tau^{\rho_n}} - Z_{\sigma^{\rho_n}}|$.

By the last statement of Lemma 3.10, a.s. $[\sigma^{\rho_n}, \tau^{\rho_n}] = [\sigma, \tau]$ and hence $\mathcal{S}_{\sigma^{\rho_n}, \tau^{\rho_n}} = \mathcal{S}_{\sigma,\tau}$ for all large enough $n \in \mathbb{N}$ and $\mathcal{F}^\rho = \bigcap_{n \in \mathbb{N}} \mathcal{F}^{\rho_n}$. By the backward martingale convergence theorem, the conditional law of $\mathcal{S}_{\sigma,\tau}$ given \mathcal{F}^ρ is that of a singly marked quantum disk with area $\tau - \sigma$ and boundary length $|Z_\tau - Z_\sigma|$. We get the same conditional law if we condition only on σ, τ , and $Z_\tau - Z_\sigma$. By [MS15c, Theorem 2.1], the curve-decorated quantum surface $(\mathcal{S}_{\sigma,\tau}, \eta'_{\mathcal{S}_{\sigma,\tau}})$ is a.s. determined by its peanosphere Brownian motion $(Z - Z_\sigma)|_{[\sigma,\tau]}$, which will be important just below.

We now allow t to vary and prove that the bubbles of $\mathcal{S}_{a,b}^0$ are conditionally independent given $Z|_{(-\infty,a]}$, $Z|_{[b,\infty)}$, and $\{Z_{\tau_{a,b}(t)} : t \in [a,b]\}$. Indeed, under this conditioning the Brownian motion increments $(Z - Z_{v_Z(s)})|_{[v_Z(s),s]}$ as s ranges over all maximal $\frac{\pi}{2}$ -cone times for Z in $[a,b]$ (Definition 2.3) are conditionally independent. By the above measurability statement, each of these Brownian motion increments a.s. determines the corresponding bubble of $\mathcal{S}_{a,b}^0$. The proposition statement follows. \square

3.4 Bubbles cut out by $\tilde{\eta}_{a,b}$ are quantum disks

Recall the future curves $\tilde{\eta}_{a,b} : [a,b] \rightarrow \tilde{\eta}'([a,b])$ associated with $(\tilde{h}, \tilde{\eta}')$ and the surface $\tilde{\mathcal{S}}_{a,b}^0$ parameterized by the bubbles it cuts out. We recall also the times $\sigma_{a,b}(t)$ and $\tau_{a,b}(t)$ from (3.3), and note that the definitions of these times do not change if we replace $\eta_{a,b}$ with $\tilde{\eta}_{a,b}$.

In this subsection we will prove the exact analog of Proposition 3.9 for the pair $(\tilde{h}, \tilde{\eta}')$.

Proposition 3.12. *Let $a, b \in \mathbb{R} \cup \{-\infty, \infty\}$ with $a < b$. If we condition on $Z|_{(-\infty,a]}$, $Z|_{[b,\infty)}$, and $\{Z_{\tau_{a,b}(t)} : t \in [a,b]\}$ then the conditional law of the bubbles of the quantum surface $\tilde{\mathcal{S}}_{a,b}^0$ is that of a collection of independent quantum disks with given areas and boundary lengths.*

As in Section 3.3, throughout most of this subsection we will fix $a, b \in \mathbb{R} \cup \{-\infty, \infty\}$ with $a < b$ and $t \in (a, b)$ and write $\tau = \tau_{a,b}(t)$ and $\sigma = \sigma_{a,b}(t)$.

The main difficulty in the proof of Proposition 3.12 is showing that the single bubble $\tilde{\mathcal{S}}_{\sigma,\tau}$ has the law of a quantum disk under certain conditioning. This will be accomplished in Lemma 3.16. Then, in the proof of Proposition 3.12 itself, we will allow t to vary.

The idea of the proof of Proposition 3.12 is to use the fact that the surfaces $\tilde{\mathcal{S}}_{r,\infty}$ and $\mathcal{S}_{r,\infty}$ agree in law for each $r \in \mathbb{R}$ (condition 1 in Theorem 2.4) and a limiting argument as r decreases to σ to relate the laws of $\tilde{\mathcal{S}}_{\sigma,\tau}$ and $\mathcal{S}_{\sigma,\tau}$; we know that the law of the latter is that of a quantum disk by Proposition 3.9.

The times σ and τ are *not* stopping times for Z or even for the filtration $\{\tilde{\mathcal{F}}_{-\infty,r}\}_{r \in \mathbb{R}}$ of Theorem 2.4 since, using the characterization from Lemma 3.1, we cannot see if a given $\frac{\pi}{2}$ -cone time for Z is maximal in $[a, b]$ without knowing some information about what happens after this time. However, as we will see just below, these times are stopping times for a larger filtration which we introduce in the following lemma.

Lemma 3.13. *For $r \in \mathbb{R}$, let $\tilde{\mathcal{F}}_{-\infty,r}$ be the σ -algebra of Theorem 2.4 and let $P_{r,\infty}$ be the function in (3.5) which encodes the ordered sequence of areas and boundary lengths of the beads of $\mathcal{S}_{r,\infty}$ (equivalently $\tilde{\mathcal{S}}_{r,\infty}$). Define σ -algebras*

$$\mathcal{G}_r := \sigma((Z - Z_r)|_{(-\infty,r]}) \vee \sigma(P_{r,\infty}) \quad \text{and} \quad \tilde{\mathcal{G}}_r := \tilde{\mathcal{F}}_{-\infty,r} \vee \sigma(P_{r,\infty}). \quad (3.9)$$

For each $r \in \mathbb{R}$, we have $\mathcal{G}_r \subset \tilde{\mathcal{G}}_r$. Furthermore, the σ -algebras $\{\mathcal{G}_r\}_{r \in \mathbb{R}}$ and $\{\tilde{\mathcal{G}}_r\}_{r \in \mathbb{R}}$ are increasing in r .

Proof. By definition, for each $r \in \mathbb{R}$ it holds that $\sigma((Z - Z_r)|_{(-\infty,r]}) \subset \tilde{\mathcal{F}}_{-\infty,r}$. Therefore, $\mathcal{G}_r \subset \tilde{\mathcal{G}}_r$.

It is clear that $\sigma((Z - Z_r)|_{(-\infty,r]}) \subset \sigma((Z - Z_{r'})|_{(-\infty,r']})$ and $\tilde{\mathcal{F}}_{-\infty,r} \subset \tilde{\mathcal{F}}_{-\infty,r'}$ for $r \leq r'$. Hence to show that our σ -algebras are increasing in r , it suffices to show that $P_{r,\infty}$ is a.s. determined by $(Z - Z_{r'})|_{(-\infty,r']}$ and $P_{r',\infty}$ for $r \leq r'$.

By definition, $P_{r,\infty}$ determines and is determined by $\{(s - r, L_s - L_r, R_s - R_r)\}_{s \in \Sigma_r}$, where Σ_r is the set of times $s \geq r$ at which L and R attain a simultaneous running infimum. Each time in $s \in \Sigma_r \cap [r', \infty)$ also belongs to $\Sigma_{r'}$, and the corresponding triple $(s - r, L_s - L_r, R_s - R_r)$ is determined by $(s - r', L_s - L_{r'}, R_s - R_{r'})$ and $(Z - Z_{r'})|_{(-\infty,r']}$. Furthermore, if $s' \in \Sigma_{r'}$ then we can determine whether $s' \in \Sigma_r$ from the triple $(s - r', L_s - L_{r'}, R_s - R_{r'})$ together with $(Z - Z_{r'})|_{(-\infty,r']}$. Hence $P_{r,\infty}$ is a.s. determined by $(Z - Z_{r'})|_{(-\infty,r']}$ and $P_{r',\infty}$ for $r \leq r'$ and we obtain the statement of the lemma. \square

The next lemma implies in particular that σ and τ are stopping times for both of the filtrations introduced in Lemma 3.13.

Lemma 3.14. *Define the σ -algebras $\{\tilde{\mathcal{G}}_r\}_{r \in \mathbb{R}}$ as in (3.9). The times σ and τ in Proposition 3.17 are stopping times for $\{\mathcal{G}_r\}_{r \in \mathbb{R}}$ (and hence also for $\{\tilde{\mathcal{G}}_r\}_{r \in \mathbb{R}}$). Furthermore, the curve-decorated quantum surface $(\mathcal{S}_{a,\tau}^0, \eta_{a,b}, \mathcal{S}_{a,\tau}^0)$ is \mathcal{G}_τ -measurable and the curve-decorated quantum surface $(\tilde{\mathcal{S}}_{a,\tau}^0, \tilde{\eta}_{a,b}, \tilde{\mathcal{S}}_{a,\tau}^0)$ is $\tilde{\mathcal{G}}_\tau$ -measurable.*

Proof. We first check that σ and τ are stopping times for $\{\mathcal{G}_r\}_{r \in \mathbb{R}}$. Recall from Lemma 3.1 that $[\sigma, \tau]$ is the maximal $\frac{\pi}{2}$ -cone interval for Z in $[a, b]$ which contains t .

If $r \notin [a, t]$, then $\{\sigma \leq r\}$ is either the empty event or the whole probability space and if $r \in [a, t]$, then $\{\sigma \leq r\}$ is a.s. the same as the event that there is a $\frac{\pi}{2}$ -cone time s for Z in $[t, b]$ such that $v_Z(s) \in [a, r]$. A $\frac{\pi}{2}$ -cone time s for Z in $[t, b]$ with $v_Z(s) \leq r$ is the same as a time $s \in [t, b]$ at which the two coordinates of $(Z - Z_r)|_{[r,\infty)}$ attain a simultaneous running infimum, and for such a $\frac{\pi}{2}$ -cone time s we have $v_Z(s) \in [a, r]$ if and only if there is a $v \in [a, r]$ such that either

$$L_v - L_r \leq L_s - L_r \quad \text{or} \quad R_v - R_r \leq R_s - R_r.$$

It is clear that $(Z - Z_r)|_{(-\infty,r]}$ and $P_{r,\infty}$ together a.s. determine whether this is the case. Thus $\{\sigma \leq r\} \in \mathcal{G}_r$, so σ is a $\{\mathcal{G}_r\}_{r \in \mathbb{R}}$ -stopping time.

Now we consider the time τ . The event $\{\tau \leq r\}$ is proper and non-empty only if $r \in [t, b]$. For such an r , we a.s. have $\tau \leq r$ if and only if the maximal $\frac{\pi}{2}$ -cone interval $[v_Z(s), s]$ for Z in $[a, r]$ which contains t is also maximal in $[a, b]$. The $\frac{\pi}{2}$ -cone interval $[v_Z(s), s]$ is maximal in $[a, b]$ if and only if there is no $\frac{\pi}{2}$ -cone time s' for Z with $s' > r$ and $v_Z(s') \in [a, v_Z(s)]$. Such a $\frac{\pi}{2}$ -cone time s' is the same as a time $s' \in [r, b]$ such that the two coordinates of $(Z - Z_r)|_{[r,\infty)}$ attain a simultaneous running infimum at time s' ; there is no $v \in [s, r]$

such that either $L_v - L_r \leq L_{s'} - L_r$ or $R_v - R_r \leq R_{s'} - R_r$; and there is a $v \in [a, v_Z(s)]$ such that either $L_v - L_r \leq L_{s'} - L_r$ or $R_v - R_r \leq R_{s'} - R_r$. We can tell whether such a time s' exists from $(Z - Z_r)|_{(-\infty, r]}$ and $P_{r, \infty}$. Hence $\{\tau \leq r\} \in \mathcal{G}_r$ and τ is a $\{\mathcal{G}_r\}_{r \in \mathbb{R}}$ -stopping time.

By [MS15c, Theorem 2.1] and Proposition 3.9, each of the bubbles $\mathcal{S}_{v_Z(s), s}$ of $\mathcal{S}_{a, \tau}^0$ is a.s. determined by the corresponding $\frac{\pi}{2}$ -cone excursion $(Z - Z_{v_Z(s)})|_{[v_Z(s), s]}$ for Z , so is a.s. determined by \mathcal{G}_r on the event $\{\tau \leq r\}$. By the peanosphere construction the equivalence class of the curve-decorated quantum surface $(\mathcal{S}_{a, \tau}^0, \eta_{a, b}, \mathcal{S}_{a, \tau}^0)$ modulo curve-preserving homeomorphisms is a.s. determined by $(Z - Z_\tau)|_{(-\infty, \tau]}$. Hence $(\mathcal{S}_{a, \tau}^0, \eta_{a, b}, \mathcal{S}_{a, \tau}^0) \in \mathcal{G}_\tau$.

The bubbles of the quantum surface $\tilde{\mathcal{S}}_{a, \tau}^0$ is parameterized by the ordered sequence of bubbles cut out by the curve $\tilde{\eta}_{a, b, \tilde{\mathcal{S}}_{a, \tau}^0}$ before time τ , so are a.s. determined by $\tilde{\mathcal{F}}_{-\infty, r}$ and τ on the event $\{\tau \leq r\}$. By condition 2 of Theorem 2.4, $(\tilde{\mathcal{S}}_{a, \tau}^0, \tilde{\eta}_{a, b, \tilde{\mathcal{S}}_{a, \tau}^0})$ a.s. differs from $(\mathcal{S}_{a, \tau}^0, \eta_{a, b}, \mathcal{S}_{a, \tau}^0)$ by a curve-preserving homeomorphism, so by the above the equivalence class of $(\mathcal{S}_{a, \tau}^0, \eta_{a, b}, \mathcal{S}_{a, \tau}^0)$ modulo curve-preserving homeomorphisms is a.s. determined by $(Z - Z_\tau)|_{(-\infty, \tau]} \in \tilde{\mathcal{G}}_\tau$. Therefore $(\tilde{\mathcal{S}}_{a, \tau}^0, \tilde{\eta}_{a, b, \tilde{\mathcal{S}}_{a, \tau}^0}) \in \tilde{\mathcal{G}}_\tau$. \square

It is easy to see from the conditions of Theorem 2.4 that for $r \in \mathbb{R}$, each of $\tilde{\mathcal{F}}_{r, \infty}$ and $\tilde{\mathcal{S}}_{r, \infty}$ is conditionally independent from the σ -algebra $\tilde{\mathcal{G}}_r$ of (3.9) given the function $P_{r, \infty}$ of (3.5). The next lemma extends this property to stopping times for this filtration (which, by Lemma 3.14, includes the times σ and τ).

Lemma 3.15. *Let T be an a.s. finite stopping time for the filtration $\{\tilde{\mathcal{G}}_r\}_{r \in \mathbb{R}}$ of Lemma 3.13. Each of $\tilde{\mathcal{F}}_{T, \infty}$ and $\tilde{\mathcal{S}}_{T, \infty}$ is conditionally independent from $\tilde{\mathcal{G}}_T$ given $P_{T, \infty}$.*

We note that Lemma 3.15 does *not* imply that the pair $(\tilde{\mathcal{F}}_{T, \infty}, \tilde{\mathcal{S}}_{T, \infty})$ is jointly conditionally independent from $\tilde{\mathcal{G}}_T$ given $P_{T, \infty}$.

Proof of Lemma 3.15. First we consider the case when $T = r$ is deterministic. By condition 1 in Theorem 2.4, each of $\tilde{\mathcal{F}}_{r, \infty}$ and $\tilde{\mathcal{S}}_{r, \infty}$ is independent from $\tilde{\mathcal{F}}_{-\infty, r}$. It is clear from the definition (3.5) that $P_{r, \infty}$ is a.s. determined by each of $\tilde{\mathcal{F}}_{r, \infty}$ and $\tilde{\mathcal{S}}_{r, \infty}$. Hence the statement of the lemma is true if T is deterministic.

Now suppose we are given a stopping time T as in the statement of the lemma and for $n \in \mathbb{N}$, let $T^n = 2^{-n} \lceil 2^n T \rceil$, so that each T^n is a stopping time for $\{\tilde{\mathcal{G}}_r\}_{r \in \mathbb{R}}$ and T^n a.s. decreases to T . For $n \in \mathbb{N}$, let

$$\mathcal{H}_{T^n} := \sigma((Z - Z_T)|_{[T, T^n]}, P_{T^n, \infty}).$$

Then $\sigma(P_{T, \infty}) \subset \mathcal{H}_{T^n} \subset \tilde{\mathcal{G}}_{T^n}$ and $\bigcap_{n=1}^{\infty} \mathcal{H}_{T^n} = \sigma(P_{T, \infty})$. By the above statement for deterministic times, it follows that the statement of the lemma is true for each T^n , i.e. for each $n \in \mathbb{N}$, each of $\tilde{\mathcal{F}}_{T^n, \infty}$ and $\tilde{\mathcal{S}}_{T^n, \infty}$ is conditionally independent from $\tilde{\mathcal{G}}_{T^n}$ given $P_{T^n, \infty}$. Since $\tilde{\mathcal{G}}_T \subset \tilde{\mathcal{G}}_{T^n}$ and $P_{T^n, \infty} \in \mathcal{H}_{T^n}$, also each of $\tilde{\mathcal{F}}_{T^n, \infty}$ and $\tilde{\mathcal{S}}_{T^n, \infty}$ is conditionally independent from $\tilde{\mathcal{G}}_T$ given \mathcal{H}_{T^n} .

For $m \geq n$, the time T^n and the σ -algebra \mathcal{H}_{T^n} are a.s. determined by $(Z - Z_{T^m})|_{[T^m, \infty)} \in \tilde{\mathcal{F}}_{T^m, \infty}$, hence \mathcal{H}_{T^n} is conditionally independent from $\tilde{\mathcal{G}}_T$ given \mathcal{H}_{T^m} . Therefore each of $\tilde{\mathcal{F}}_{T^n, \infty}$ and $\tilde{\mathcal{S}}_{T^n, \infty}$ is conditionally independent from $\tilde{\mathcal{G}}_T$ given \mathcal{H}_{T^m} . Taking a limit as $m \rightarrow \infty$ (n fixed) and applying the backward martingale convergence theorem to $\mathbb{P}[A \cap B | \mathcal{H}_{T^m}]$ for events $A \in \tilde{\mathcal{F}}_{T^n, \infty}$ or $A \in \sigma(\tilde{\mathcal{S}}_{T^n, \infty})$ and $B \in \tilde{\mathcal{G}}_T$ shows that each of $\tilde{\mathcal{F}}_{T^n, \infty}$ and $\tilde{\mathcal{S}}_{T^n, \infty}$ is conditionally independent from $\tilde{\mathcal{G}}_T$ given $P_{T, \infty}$.

By the continuity of Z , the $\tilde{\mathcal{F}}_{T^n, \infty}$ -measurable functions which equal $Z - Z_{T^n}$ on $[T^n, \infty)$ and 0 on $(-\infty, T^n]$ converge uniformly to $(Z - Z_T)|_{[T, \infty)}$ on $[T, \infty)$. If $[v_Z(s), s]$ is a $\pi/2$ -cone interval contained in (T, ∞) , then a.s. $[v_Z(s), s]$ is contained in $[T^n, \infty)$ for large enough n . On the other hand, if $a < t < b$ are rational times such that $\sigma_{a, b}(t) = T$ then since the maximal $\pi/2$ -cone interval $[\sigma_{a, b}(t), \tau_{a, b}(t)]$ can be approximated arbitrarily closely by smaller $\pi/2$ -cone intervals contained in $[\sigma_{a, b}(t), \tau_{a, b}(t)]$ and since $\tilde{\eta}'$ is continuous, the $\tilde{\mathcal{F}}_{T^n, \infty}$ -measurable quantum surfaces $\tilde{\mathcal{S}}_{\sigma_{a \vee T^n, b}(t), \tau_{a \vee T^n, b}(t)}$ (each equipped with an interior marked point sampled uniformly from its area measure) converge a.s. in the topology of Section 2.2.6 to $\tilde{\mathcal{S}}_{\sigma_{a, b}(t), \tau_{a, b}(t)}$. From these convergence statements and conditional independence of $\tilde{\mathcal{F}}_{T^n, \infty}$ and $\tilde{\mathcal{G}}_T$ given $P_{T, \infty}$, we infer that $\tilde{\mathcal{F}}_{T, \infty}$ is conditionally independent from $\tilde{\mathcal{G}}_T$ given $P_{T, \infty}$.

Furthermore, by continuity of $\tilde{\eta}'$ a.s. $\tilde{\mathcal{S}}_{T^n, \infty} \rightarrow \tilde{\mathcal{S}}_{T, \infty}$ with respect to the topology induced by the metric (2.7) of Section 2.2.6 (where here we sample a uniformly random point from the area measure of each bead of $\tilde{\mathcal{S}}_{T, \infty}$

to get an interior marked point). Therefore $\tilde{\mathcal{S}}_{T,\infty}$ is conditionally independent from $\tilde{\mathcal{G}}_T$ given $P_{T,\infty}$ and we conclude. \square

The following lemma tells us the conditional law of the single bubble $\tilde{\mathcal{S}}_{\sigma,\tau}$ of $\tilde{\mathcal{S}}_{a,b}^0$ given the previous bubbles as well as the segments of the Brownian motion Z corresponding to the complement of the bubble. This lemma is the key ingredient in the proof of Proposition 3.12.

Lemma 3.16. *Almost surely, the conditional law of the quantum surface $\tilde{\mathcal{S}}_{\sigma,\tau}$ parameterized by the bubble $\tilde{\eta}'([\sigma,\tau])$ cut out by $\tilde{\eta}_{a,b}$ given the σ -algebra $\tilde{\mathcal{G}}_\sigma$ of (3.9) and $(Z - Z_\tau)|_{[\tau,\infty)}$ is that of a singly marked quantum disk with area $\tau - \sigma$ and boundary length $|Z_\tau - Z_\sigma|$.*

Proof. For $n \in \mathbb{N}$, let $\sigma^n = 2^{-n}\lceil 2^n\sigma \rceil$. By Lemma 3.14, σ is a stopping time for the filtration $\{\tilde{\mathcal{G}}_r\}_{r \in \mathbb{R}}$, hence the same is the case for each σ^n . By condition 1 in Theorem 2.4, for each $n \in \mathbb{N}$ the conditional law of $\tilde{\mathcal{S}}_{\sigma^n,\infty}$ given $\tilde{\mathcal{G}}_{\sigma^n}$ is that of a collection of independent beads of a $\frac{3\gamma}{2}$ -quantum wedge with given areas and left/right quantum boundary lengths.

If $t > \sigma^n$, let $\tilde{\mathcal{D}}^n$ be the bead of $\tilde{\mathcal{S}}_{\sigma^n,\infty}$ which contains $\tilde{\eta}'_{\tilde{\mathcal{S}}_{\sigma^n,\infty}}(t)$, equivalently the first bead of $\tilde{\mathcal{S}}_{\sigma^n,\infty}$ with the property that the total quantum mass of the previous beads is at least $t - \sigma^n$ (including $\tilde{\mathcal{D}}^n$ itself). If $t \leq \sigma^n$ let $\tilde{\mathcal{D}}^n$ be the trivial one-point quantum surface. Also let τ^n be the time at which $\tilde{\eta}'_{\tilde{\mathcal{S}}_{\sigma^n,\infty}}$ finishes filling in this bead, equivalently $\tau^n - \sigma^n$ is the quantum mass of $\tilde{\mathcal{D}}^n$.

The conditional law of $\tilde{\mathcal{D}}^n$ given $\tilde{\mathcal{G}}_{\sigma^n}$ is that of a single bead of a $\frac{3\gamma}{2}$ -quantum wedge with given area and left/right boundary lengths. This conditional law depends only on the realization of $P_{\sigma^n,\infty}$, so in particular $\tilde{\mathcal{D}}^n$ is conditionally independent from $\tilde{\mathcal{G}}_{\sigma^n}$ given $P_{\sigma^n,\infty}$.

We will now compare $\tilde{\mathcal{D}}^n$ to the analogous object defined in terms of the pair (h, η') . For $n \in \mathbb{N}$, let \mathcal{D}^n be the bead of $\mathcal{S}_{\sigma^n,\infty}$ which contains $\eta'_{\mathcal{S}_{\sigma^n,\infty}}(t)$ (or a point if $t \leq \sigma^n$). By the conclusion of the preceding paragraph applied in the case when $(\tilde{h}, \tilde{\eta}') = (h, \eta')$, we find that the conditional law of \mathcal{D}^n given σ^n and $P_{\sigma^n,\infty}$ is that of a single bead of a $\frac{3\gamma}{2}$ -quantum wedge with given area and left/right boundary lengths (here we note that the definitions of σ^n and $P_{\sigma^n,\infty}$ depend only on (h, η') , i.e. the conditional laws of $\tilde{\mathcal{D}}^n$ and \mathcal{D}^n given $P_{\sigma^n,\infty}$ are the same. Therefore,

$$\left(\tilde{\mathcal{D}}^n, P_{\sigma^n,\infty}\right) \stackrel{d}{=} (\mathcal{D}^n, P_{\sigma^n,\infty}), \quad \forall n \in \mathbb{N}. \quad (3.10)$$

By continuity of $\tilde{\eta}'$, as $n \rightarrow \infty$ the sub-domain of \mathbb{C} with two marked boundary points which parameterizes $\tilde{\mathcal{D}}^n$ converges a.s. in the Caratheodory sense to $(\tilde{\eta}'([\sigma,\tau]), \tilde{\eta}'(\tau))$. Hence $\tilde{\mathcal{D}}^n$ converges a.s. in the topology of Section 2.2.6 to $\tilde{\mathcal{S}}_{\sigma,\tau}$ (where here we sample a uniformly random point from the area measure on $\tilde{\mathcal{S}}_{\sigma,\tau}$ to get an interior marked point). Furthermore, since $\sigma^n \rightarrow \sigma$ a.s., by continuity of Z a.s. $P_{\sigma^n,\infty} \rightarrow P_{\sigma,\infty}$ uniformly. Therefore, a.s.

$$\left(\tilde{\mathcal{D}}^n, P_{\sigma^n,\infty}\right) \rightarrow \left(\tilde{\mathcal{S}}_{\sigma,\tau}, P_{\sigma,\infty}\right). \quad (3.11)$$

Similarly, a.s.

$$(\mathcal{D}^n, P_{\sigma^n,\infty}) \rightarrow (\mathcal{S}_{\sigma,\tau}, P_{\sigma,\infty}). \quad (3.12)$$

The convergence (3.12) also occurs in law, so by (3.10) and (3.11),

$$\left(\tilde{\mathcal{S}}_{\sigma,\tau}, P_{\sigma,\infty}\right) \stackrel{d}{=} (\mathcal{S}_{\sigma,\tau}, P_{\sigma,\infty}). \quad (3.13)$$

In particular, Proposition 3.9 implies that the conditional law of $\tilde{\mathcal{S}}_{\sigma,\tau}$ given its area and boundary length (which are necessarily equal to $\tau - \sigma$ and $|Z_\tau - Z_\sigma|$, respectively) is that of a singly marked quantum disk with given area and boundary length. Furthermore, since $P_{\sigma,\infty}$ is a.s. determined by σ , $Z_\tau - Z_\sigma$, and $(Z - Z_\tau)|_{[\tau,\infty)}$, Proposition 3.9 implies that $\tilde{\mathcal{S}}_{\sigma,\tau}$ is conditionally independent from $P_{\sigma,\infty}$ given $\tau - \sigma$ and $|Z_\tau - Z_\sigma|$. Note that $\tau - \sigma$ and $|Z_\tau - Z_\sigma|$ are a.s. determined by $P_{\sigma,\infty}$ since these quantities give the area and left/right boundary lengths of the first bead of $\tilde{\mathcal{S}}_{\sigma,\infty}$.

We will now argue that we get the same conditional law if we further condition on $\tilde{\mathcal{G}}_\sigma$ and $(Z - Z_\tau)|_{[\tau,\infty)}$. We showed above that $\tilde{\mathcal{D}}^n$ is conditionally independent from $\tilde{\mathcal{G}}_{\sigma^n}$ given $P_{\sigma^n,\infty}$ for each $n \in \mathbb{N}$. Since $\tilde{\mathcal{G}}_\sigma \subset \tilde{\mathcal{G}}_{\sigma^n}$

for each $n \in \mathbb{N}$ (Lemma 3.13), $\tilde{\mathcal{D}}^n$ is conditionally independent from $\tilde{\mathcal{G}}_\sigma$ given $P_{\sigma^n, \infty}$. The function $P_{\sigma^n, \infty}$ is determined by $(Z - Z_\sigma)|_{[\sigma, \infty)}$ so by Lemma 3.15 is conditionally independent from $\tilde{\mathcal{G}}_\sigma$ given $P_{\sigma, \infty}$. Hence the conditional law of $\tilde{\mathcal{D}}^n$ given $\tilde{\mathcal{G}}_\sigma$ can be sampled from as follows.

1. Sample $P_{\sigma^n, \infty}$ from its conditional law given $\tilde{\mathcal{G}}_\sigma$ (which depends only on $P_{\sigma, \infty}$).
2. Sample $\tilde{\mathcal{D}}^n$ from its conditional law given $P_{\sigma^n, \infty}$.

In particular, this conditional law depends only on $P_{\sigma, \infty}$ so $\tilde{\mathcal{D}}^n$ is conditionally independent from $\tilde{\mathcal{G}}_\sigma$ given $P_{\sigma, \infty}$. Since this holds for each $n \in \mathbb{N}$, it follows from (3.11) that $\tilde{\mathcal{S}}_{\sigma, \tau}$ is conditionally independent from $\tilde{\mathcal{G}}_{\sigma, \infty}$ given $P_{\sigma, \infty}$.

By Lemma 3.15 and since τ is a $\{\tilde{\mathcal{G}}_r\}_{r \in \mathbb{R}}$ -stopping time (Lemma 3.14), the Brownian motion segment $(Z - Z_\tau)|_{[\tau, \infty)}$ is conditionally independent from $\tilde{\mathcal{G}}_\tau$ given $P_{\tau, \infty}$. Since the two coordinates of $(Z - Z_\sigma)|_{[\sigma, \infty)}$ attain a simultaneous running infimum at time τ , it follows from (3.5) that $P_{\tau, \infty} = P_{\sigma, \infty}|_{[\tau, \infty)}$. The time τ is determined by $P_{\sigma, \infty}$ since $\tau - \sigma$ is the length of the first interval of times on which $P_{\tau, \infty}$ is constant. Therefore, $P_{\tau, \infty}$ is a.s. determined by $P_{\sigma, \infty}$ so $(Z - Z_\tau)|_{[\tau, \infty)}$ is conditionally independent from $\tilde{\mathcal{G}}_\tau$ given $P_{\sigma, \infty}$. It is clear that $\tilde{\mathcal{G}}_\sigma$ and $\sigma(\tilde{\mathcal{S}}_{\tau, \sigma})$ are contained in $\tilde{\mathcal{G}}_\tau$. By combining this with the preceding paragraph, we see that $\tilde{\mathcal{S}}_{\sigma, \tau}$ is conditionally independent from $\tilde{\mathcal{G}}_\sigma$ and $(Z - Z_\tau)|_{[\tau, \infty)}$ given $P_{\sigma, \infty}$.

Hence the conditional law of $\tilde{\mathcal{S}}_{\sigma, \tau}$ given $\tilde{\mathcal{G}}_\sigma$ and $(Z - Z_\tau)|_{[\tau, \infty)}$ is the same as its conditional law given only $P_{\sigma, \infty}$. The discussion just after (3.13) implies that this conditional law is that of a singly marked quantum disk with area $\tau - \sigma$ and boundary length $Z_\sigma - Z_\tau$. \square

Proof of Proposition 3.12. Fix times $a < t_1 < \dots < t_n < b$. By Lemma 3.16, for each $k \in [1, n]_{\mathbb{Z}}$ the conditional law of the bubble $\tilde{\mathcal{S}}_{\sigma_{a,b}(t_k), \tau_{a,b}(t_k)}$ given $\tilde{\mathcal{G}}_{\sigma_{a,b}(t_k)}$ and $(Z - Z_{\tau_{a,b}(t_k)})|_{[\tau_{a,b}(t_k), \infty)}$ is that of a singly marked quantum disk with area $\tau_{a,b}(t_k) - \sigma_{a,b}(t_k)$ and boundary length $|Z_{\tau_{a,b}(t_k)} - Z_{\sigma_{a,b}(t_k)}|$.

By definition, $\tilde{\mathcal{G}}_{\sigma_{a,b}(t_k)}$ and $(Z - Z_{\tau_{a,b}(t_k)})|_{[\tau_{a,b}(t_k), \infty)}$ together a.s. determine Z_s for $s \in \mathbb{R} \setminus [\sigma_{a,b}(t_k), \tau_{a,b}(t_k)]$ (here we note that $\tilde{\mathcal{S}}_{\sigma_{a,b}(t_k), \tau_{a,b}(t_k)}$ is the same as the first bead of $\tilde{\mathcal{S}}_{\sigma_{a,b}(t_k), \infty}$, so its area and boundary length are $\tilde{\mathcal{G}}_{\sigma_{a,b}(t_k)}$ -measurable). In particular, $\tilde{\mathcal{G}}_{\sigma_{a,b}(t_k)}$ and $(Z - Z_{\tau_{a,b}(t_k)})|_{[\tau_{a,b}(t_k), \infty)}$ together a.s. determine $Z|_{(-\infty, a]}$, $Z|_{[b, \infty)}$, and $\{Z_{\tau_{a,b}(t)} : t \in [a, b]\}$. Furthermore, the σ -algebra $\tilde{\mathcal{G}}_{\sigma_{a,b}(t_k)}$ a.s. determines the previous bubbles $\tilde{\mathcal{S}}_{\sigma_{a,b}(t_j), \tau_{a,b}(t_j)}$ for $j \in [1, k-1]_{\mathbb{Z}}$ which are not equal to $\tilde{\mathcal{S}}_{\sigma_{a,b}(t_k), \tau_{a,b}(t_k)}$.

The preceding two paragraphs together imply that the conditional law of $\tilde{\mathcal{S}}_{\sigma_{a,b}(t_k), \tau_{a,b}(t_k)}$ given $Z|_{(-\infty, a]}$, $Z|_{[b, \infty)}$, $\{Z_{\tau_{a,b}(t)} : t \in [a, b]\}$, and $\tilde{\mathcal{S}}_{\sigma_{a,b}(t_j), \tau_{a,b}(t_j)}$ for $j \in [1, k-1]_{\mathbb{Z}}$ such that this bubble is not equal to $\tilde{\mathcal{S}}_{\sigma_{a,b}(t_k), \tau_{a,b}(t_k)}$ is that of a singly marked quantum disk with area $\tau_{a,b}(t_k) - \sigma_{a,b}(t_k)$ and boundary length $|Z_{\tau_{a,b}(t_k)} - Z_{\sigma_{a,b}(t_k)}|$. This holds for each $k \in [1, n]_{\mathbb{Z}}$, so we infer that the conditional law of the distinct bubbles in $\{\tilde{\mathcal{S}}_{\sigma_{a,b}(t_k), \tau_{a,b}(t_k)}\}_{k \in [1, n]_{\mathbb{Z}}}$ given $Z|_{(-\infty, a]}$, $Z|_{[b, \infty)}$, and $\{Z_{\tau_{a,b}(t)} : t \in [a, b]\}$ is that of a collection of independent singly marked quantum disks with given areas and boundary lengths. Since the number of times n we are considering can be made arbitrarily large, we conclude. \square

3.5 Joint laws of intermediate surfaces

In this subsection we will prove an equality in law between certain quantum surfaces defined in terms of $(\tilde{h}, \tilde{\eta}')$ and their counterparts defined in terms of (h, η') which builds on the results of the previous two subsections. This result will be a key input in the “curve-swapping” argument used in Section 5 to prove Theorem 2.4. Throughout, we fix $a, t, b \in \mathbb{R}$ with $a < t < b$ and we let $\sigma = \sigma_{a,b}(t)$ and $\tau = \tau_{a,b}(t)$ be as in (3.3). We recall the future surface $\mathcal{S}_{a, \infty}$ and the surface $\mathcal{S}_{a, \tau}^0$ parameterized by the bubbles cut out by $\eta_{a, \tau} = \eta_{a, b}|_{[0, \tau]}$ from Section 3.1 and their counterparts with $(\tilde{h}, \tilde{\eta}')$ in place of (h, η') .

The main goal of this subsection is to prove the following.

Proposition 3.17. *In the setting described just above,*

$$\left(\tilde{\mathcal{S}}_{a, \tau}^0, \tilde{\eta}_{a, b}, \tilde{\mathcal{S}}_{a, \tau}^0, \tilde{\mathcal{S}}_{\tau, \infty} \right) \stackrel{d}{=} \left(\mathcal{S}_{a, \tau}^0, \eta_{a, b}, \mathcal{S}_{a, \tau}^0, \mathcal{S}_{\tau, \infty} \right). \quad (3.14)$$

We emphasize that $\tilde{\mathcal{S}}_{a,b}^0$ for $a < b$ encodes only the quantum surfaces parameterized by the bubbles cut out by $\tilde{\eta}'$ in the time interval $[a, b]$ and the topology of how these quantum surfaces are glued together but not the whole quantum surface structure of $\tilde{\mathcal{S}}_{a,b}$.

The proof of Proposition 3.17 proceeds by way of two lemmas, which are in turn consequences of the results of Sections 3.3 and 3.4. The first lemma is essentially an extension of Proposition 3.12.

Lemma 3.18. *We have the equalities in law*

$$\begin{aligned} & \left(\tilde{\mathcal{S}}_{a,b}^0, \tilde{\eta}_{a,b}, \tilde{\mathcal{S}}_{a,b}^0, Z|_{\mathbb{R} \setminus [a,b]}, \{Z_{\tau_{a,b}(s)} : s \in [a,b]\} \right) \\ & \stackrel{d}{=} \left(\mathcal{S}_{a,b}^0, \eta_{a,b}, \mathcal{S}_{a,b}^0, Z|_{\mathbb{R} \setminus [a,b]}, \{Z_{\tau_{a,b}(s)} : s \in [a,b]\} \right) \end{aligned} \quad (3.15)$$

and

$$\left(\tilde{\mathcal{S}}_{a,\tau}^0, \tilde{\eta}_{a,b}, \tilde{\mathcal{S}}_{a,\tau}^0, P_{\tau,\infty} \right) \stackrel{d}{=} \left(\mathcal{S}_{a,\tau}^0, \eta_{a,b}, \mathcal{S}_{a,\tau}^0, P_{\tau,\infty} \right). \quad (3.16)$$

Proof. By Propositions 3.9 and 3.12, the conditional laws of the bubbles of $\mathcal{S}_{a,b}^0$ and those of $\tilde{\mathcal{S}}_{a,b}^0$ given $Z|_{\mathbb{R} \setminus [a,b]}$ and $\{Z_{\tau_{a,b}(s)} : s \in [a,b]\}$ a.s. agree. By condition 2 in Theorem 2.4, the equivalence classes of the curve-decorated quantum surfaces $(\tilde{\mathcal{S}}_{a,b}^0, \tilde{\eta}_{a,b}, \tilde{\mathcal{S}}_{a,b}^0)$ and $(\mathcal{S}_{a,b}^0, \eta_{a,b}, \mathcal{S}_{a,b}^0)$ are a.s. given by the same deterministic functional of $\{Z_{\tau_{a,b}(s)} : s \in [a,b]\}$ (c.f. [DMS14, Figure 1.18, Line 3]). Hence (3.15) holds.

We now deduce (3.16) from (3.15). By (3.3) (resp. its analog with $\tilde{\eta}_{a,b}$ in place of $\eta_{a,b}$), the time $\tau = \tau_{a,b}(t)$ is the infimum of the times $s \geq t$ such that $\eta_{a,b}(s) \neq \eta_{a,b}(t)$ (resp. $\tilde{\eta}_{a,b}(s) \neq \tilde{\eta}_{a,b}(t)$), so can be obtained by applying the same deterministic functional to either the left or the right side of (3.15). The quantum surface $\tilde{\mathcal{S}}_{a,\tau}^0$ are parameterized by the bubbles of $\tilde{\mathcal{S}}_{a,b}^0$ filled in by $\tilde{\eta}'$ before time $\tau = \tau_{a,b}(t)$, so is a.s. determined by the left side of (3.15). Similarly, $\mathcal{S}_{a,b}^0$ is a.s. determined by the right side of (3.15), in the same deterministic manner. None of the times \underline{T}_τ or \overline{T}_τ appearing in the definition 3.5 of $P_{\tau,\infty}$ is contained in one of the intervals $(\sigma_{a,b}(s), \tau_{a,b}(s))$ for $s \in [a,b]$. Therefore, $P_{\tau,\infty}$ is a.s. determined by $Z|_{[b,\infty)}$ and $\{Z_{\tau_{a,b}(s)} : s \in [a,b]\}$. We therefore obtain (3.16) by applying the same deterministic functional to both sides of (3.15). \square

The second lemma we need for the proof of Proposition 3.17 is an extension of Lemma 3.15.

Lemma 3.19. *Let T be a stopping time for the filtration $\{\mathcal{G}_r\}_{r \in \mathbb{R}}$ of Lemma 3.13. The following conditional laws a.s. coincide.*

1. *The conditional law of $\tilde{\mathcal{S}}_{T,\infty}$ given $\tilde{\mathcal{G}}_T$.*
2. *The conditional law of $\mathcal{S}_{T,\infty}$ given \mathcal{G}_T .*
3. *The conditional law of either $\mathcal{S}_{T,\infty}$ or $\tilde{\mathcal{S}}_{T,\infty}$ given $P_{T,\infty}$.*

Proof. The time T is also a stopping time for the larger filtration $\{\tilde{\mathcal{G}}_r\}_{r \in \mathbb{R}}$ so by Lemma 3.15, the conditional law of $\tilde{\mathcal{S}}_{T,\infty}$ given $\tilde{\mathcal{G}}_T$ is the same as its conditional law given only $P_{T,\infty}$. By the same lemma, applied in the special case when $(\tilde{h}, \tilde{\eta}') = (h, \eta')$, we find that the conditional law of $\mathcal{S}_{T,\infty}$ given \mathcal{G}_T is the same as its conditional law given only $P_{T,\infty}$. Hence it suffices to show that

$$(\tilde{\mathcal{S}}_{T,\infty}, P_{T,\infty}) \stackrel{d}{=} (\mathcal{S}_{T,\infty}, P_{T,\infty}). \quad (3.17)$$

In the case when $T = r$ is deterministic, the conditions in Theorem 2.4 imply that both sides of (3.17) have the same law as a $\frac{3\gamma}{2}$ -quantum wedge together with the function which gives the areas and left/right boundary lengths of its beads. Hence (3.17) holds in this case. In the case when T takes on only countably many possible values, we infer from the deterministic case that the conditional law of $\tilde{\mathcal{S}}_{T,\infty}$ given $P_{T,\infty}$ is that of a sequence of beads of a $\frac{3\gamma}{2}$ -quantum wedge with areas and left/right boundary lengths specified by $P_{T,\infty}$; and the same holds for $\mathcal{S}_{T,\infty}$. Hence (3.17) holds in the case when T takes on only countably many values.

In general, we let $T^n = 2^{-n} \lceil 2^n T \rceil$, so that each T^n is a stopping time for $\{\mathcal{G}_r\}_{r \in \mathbb{N}}$ and $T^n \rightarrow T$ a.s. By taking the limits of both sides of (3.17) with T^n in place of T , we obtain (3.17) in general. \square

Proof of Proposition 3.17. By Lemma 3.14, τ is a stopping time for the filtration $\{\mathcal{G}_r\}_{r \in \mathbb{R}}$ of Lemma 3.13. Furthermore, the triples in (3.16) from Lemma 3.18 satisfy

$$\left(\tilde{\mathcal{S}}_{a,\tau}^0, \tilde{\eta}_{a,b,\tilde{\mathcal{S}}_{a,\tau}^0}, P_{\tau,\infty}\right) \in \tilde{\mathcal{G}}_\tau \quad \text{and} \quad \left(\mathcal{S}_{a,\tau}^0, \eta_{a,b,\mathcal{S}_{a,\tau}^0}, P_{\tau,\infty}\right) \in \mathcal{G}_\tau.$$

Hence Lemma 3.19 implies that the conditional law of $\tilde{\mathcal{S}}_{\tau,\infty}$ given the left side of (3.16) a.s. coincides with the conditional law of $\mathcal{S}_{\tau,\infty}$ given the right side of (3.16) (and this conditional law depends only on $P_{\tau,\infty}$). By combining this with (3.16), we obtain (3.14). \square

4 Diameter estimate for space-filling SLE segments

In this section we will prove an estimate (Proposition 4.1 just below) which will be used to bound the diameters of the hulls (and thereby, via [DMS14, Lemma 10.5] the distortion of the conformal maps) in the curve-swapping argument used in the proof of Theorem 2.4. In Section 4.4, we will also introduce a regularity event which is similar to the one used in [DMS14, Section 10.4.2] which will enable us to apply our estimate.

Before stating our estimate, we first define a particular embedding of a γ -quantum cone which is also used in [DMS14, Section 10.4.2]. Fix once and for all a smooth non-negative bump function ϕ supported on $(-1/100, 0]$ with total integral one. Suppose h is an embedding of a γ -quantum cone in $(\mathbb{C}, 0, \infty)$. For $r \geq 0$, let $h_r(0)$ be the circle average of h over $\partial B_r(0)$. For $t \in \mathbb{R}$, define

$$V_t := \frac{1}{\gamma} \log \mathbb{E}[\mu_h(B_{e^{-t}}(0)) | h_{e^{-t}}(0)]. \quad (4.1)$$

For $C \in \mathbb{R}$, we say that the embedding h is a C -smooth canonical description of $(\mathbb{C}, h, 0, \infty)$ if

$$\inf \left\{ s \in \mathbb{R} : \int_{-1/100}^0 V_t \phi(s-t) dt = C \right\} = 0. \quad (4.2)$$

A smooth canonical description is not unique: applying a rotation gives another smooth canonical description. Similarly to the circle average embedding, if h is a C -smooth canonical description then $h|_{\mathbb{D}}$ agrees in law with the restriction to \mathbb{D} of $h^0 - \gamma \log |\cdot| + \chi$, where h^0 is a whole-plane GFF normalized so that its circle average over $\partial \mathbb{D}$ is zero and χ is a random variable whose absolute value is stochastically dominated by a constant (depending only on C) plus the modulus of centered Gaussian random variable with variance bounded above by a universal constant.

Our reason for considering the smooth canonical description is that this is the embedding used in [DMS14, Section 10] to prove that a certain regularity condition for the γ -quantum cone holds whenever we “swap out” part of the surface and replace it with another surface (see Section 4.4). This estimate will be important in Section 5. The advantage of the smooth canonical description over the circle average embedding is that it behaves more nicely than the circle average embedding under conformal maps between subsets of \mathbb{C} which are not affine.

The main result of this section is the following proposition.

Proposition 4.1. *Let $\kappa' \in (4, 8)$ and $\gamma = 4/\sqrt{\kappa'}$. Let $(\mathbb{C}, h, 0, \infty)$ be a γ -quantum cone and suppose that h is either a circle-average embedding or a C -smooth canonical description for some $C \in \mathbb{R}$. Also let η' be a whole-plane space-filling $\text{SLE}_{\kappa'}$ from ∞ to ∞ independent from h and parameterized by γ -quantum mass with respect to h . Let $a, b \in \mathbb{R}$ with $a < b$ and let S be a deterministic closed subset of $\mathbb{D} \setminus \{0\}$.*

Recall the chordal $\text{SLE}_{\kappa'}$ -type curve $\eta_{a,b} : [a, b] \rightarrow \eta'([a, b])$ from Section 3. For $n \in \mathbb{N}$ and $k \in [1, n]_{\mathbb{Z}}$, let $t_{n,k} := a + \frac{k}{n}(b-a)$ and let

$$G_{n,k} := \{\eta'([t_{n,k-1}, t_{n,k}]) \cap \eta_{a,b} \neq \emptyset\} \cap \{\eta_{a,b}([t_{n,k-1}, t_{n,k}]) \subset S\}.$$

There is an $\alpha = \alpha(\kappa') > 0$ such that for $n \in \mathbb{N}$,

$$\sum_{k=1}^n \mathbb{E}[\text{diam}(\eta'([t_{n,k-1}, t_{n,k}]))^2 \mathbb{1}_{G_{n,k}}] \leq n^{-\alpha+o_n(1)} \quad (4.3)$$

at a rate depending on C, a, b, S , and κ' .

The important point in Proposition 4.1 is that the right side of (4.3) decays like a positive power of n . For our purposes the particular power of n does not matter. However, our proof will show that one can take

$$\alpha = \frac{2(\gamma^2 - 2)}{\gamma^2(4 + 3\gamma^2 + 2\sqrt{2\gamma^4 + 8\gamma^2 - 4})} = \frac{(8 - \kappa')\kappa'}{16(12 + \kappa' + \sqrt{32(4 + \kappa') - (\kappa')^2})}. \quad (4.4)$$

To prove Proposition 4.1, we will first prove an upper bound for $\mathbb{E}[\text{Area}(B_\epsilon(\eta_{a,b} \cap S))]$, where here Area denotes two-dimensional Lebesgue measure. To do this we will use a version of the KPZ formula (Proposition 4.2) which relates the expected Lebesgue measure of the ϵ -neighborhood of a subset of \mathbb{C} which is independent from h (but not from η') to the expected number of ϵ -length intervals needed to cover its pre-image under η' . This statement is similar to the main result of [GHM15], but we prove only a one-sided bound and our estimate concerns expected areas rather than a.s. Hausdorff dimensions.

In Section 4.2, we will use our KPZ formula to reduce the problem of estimating $\mathbb{E}[\text{Area}(B_\epsilon(\eta_{a,b} \cap S))]$ to the problem of estimating the expected number of ϵ -length intervals needed to cover a certain subset of \mathbb{R} described in terms of the peanosphere Brownian motion Z . We will then estimate this quantity via a straightforward Brownian motion calculation.

In Section 4.3, we will deduce the diameter estimate (4.3) from the aforementioned estimate for $\mathbb{E}[\text{Area}(B_\epsilon(\eta_{a,b} \cap S))]$. This is accomplished by means of an estimate (stated as Lemma 4.3 below) from [GHM15] which tells us that the diameter of a space-filling SLE segment is very unlikely to be larger than the square of its area, up to an $o(1)$ error in the exponent.

In Section 4.4, we will introduce a regularity event (defined in terms of the C -smooth canonical description) which will allow us to apply the estimate of Proposition 4.1 in Section 5.

4.1 KPZ formula for expected areas

Let $\kappa' > 4$ and $\gamma = 4/\sqrt{\kappa'}$. Let $(\mathbb{C}, h, 0, \infty)$ be a γ -quantum cone and let η' be an independent whole-plane space-filling $\text{SLE}_{\kappa'}$ from ∞ to ∞ parameterized by γ -quantum mass with respect to h . In this subsection, we will prove a KPZ-type formula which gives an upper bound for the expected Lebesgue measure of the ϵ -neighborhood of a set $X \subset \mathbb{C}$ which is independent from h (but not necessarily from η') in terms of the number of ϵ -length intervals needed to cover a set $\hat{X} \subset \mathbb{R}$ with $\eta'(\hat{X}) = X$. We emphasize that, unlike most of the results of this paper, the results of this subsection are valid for any $\kappa' > 4$ (and hence any $\gamma \in (0, 2)$) rather than only for $\kappa' \in (4, 8)$ ($\gamma \in (\sqrt{2}, 2)$).

Proposition 4.2. *Let $\kappa' > 4$ and $\gamma = 4/\sqrt{\kappa'}$. Let $(\mathbb{C}, h, 0, \infty)$ be a γ -quantum cone and suppose that h is either a circle-average embedding or a C -smooth canonical description for some $C \in \mathbb{R}$. Also let η' be a whole-plane space-filling $\text{SLE}_{\kappa'}$ from ∞ to ∞ independent from h and parameterized by γ -quantum mass with respect to h . Let S be a deterministic closed subset of $\mathbb{D} \setminus \{0\}$ and let X be a random subset of S such that the pair consisting of the set X and the curve η' , viewed modulo monotone re-parameterization, is independent from h (e.g., X could be determined by η' , viewed modulo monotone-reparameterization).*

Let $\hat{X} \subset \mathbb{R}$ be a random set such that $X \subset \eta'(\hat{X})$ a.s. For $\epsilon > 0$, let N_ϵ be the number of intervals of length ϵ needed to cover $(\eta')^{-1}(\hat{X})$. Suppose that for some $\beta \in [0, 1]$, it holds that

$$\limsup_{\epsilon \rightarrow 0} \frac{\log \mathbb{E}[N_\epsilon]}{\log \epsilon^{-1}} \leq \beta. \quad (4.5)$$

Then

$$\limsup_{\epsilon \rightarrow 0} \frac{\log \mathbb{E}[\text{Area}(B_\epsilon(X))]}{\log \epsilon^{-1}} \leq \left(2 + \frac{\gamma^2}{2}\right)\beta - \frac{\gamma^2}{2}\beta^2 - 2. \quad (4.6)$$

Proposition 4.2 is a one-sided version of the KPZ formula [KPZ88, DS11] which gives an upper bound for the expected Minkowski content of X in terms of the expected Minkowski content of $(\eta')^{-1}(X)$. The proposition is closely related to [GHM15, Theorem 1.1] (which gives an analogous relation for Hausdorff dimension, with an equality in place of an inequality) and [GHS16a, Proposition 3.4] (which gives an upper bound for the Minkowski content of $(\eta')^{-1}(X)$ in terms of the Minkowski content of X).

Proposition 4.2 and the aforementioned related results are especially useful for computing dimensions and exponents for sets defined in terms of SLE. The reason for this is that for many SLE sets of interest, the set

$(\eta')^{-1}(X)$ admits a simple description in terms of the peanosphere Brownian motion Z . This reduces an SLE computation to a Brownian motion computation which is often much easier. We will apply Proposition 4.2 in Section 4.2 below to prove an upper bound for the expected area of the ϵ -neighborhood of $\eta_{a,b} \cap S$, where $\eta_{a,b}$ is the curve appearing in Proposition 4.1. This bound, in turn, will be the key input in the proof of Proposition 4.1.

The proof of the KPZ relation Proposition 4.2 is similar in spirit to the proof of the upper bound for the Hausdorff dimension of X in [GHM15, Theorem 1.1]. We first state a basic preliminary estimate for space-filling $\text{SLE}_{\kappa'}$ (Lemma 4.3) and a basic estimate for the γ -LQG measure (Lemma 4.4) which follow from estimates in [GHM15]. We then consider for $\alpha > 0$ the set $X_\epsilon^\alpha \subset X$ which is, roughly speaking, the set of points in X which are contained in a segment of η' with diameter of order ϵ and μ_h -mass of order $\epsilon^{2+\frac{\gamma^2}{2}-\alpha}$. The set X_ϵ^α is typically much smaller than X since typically the μ_h -mass of a segment of η' with diameter of order ϵ is of order $\epsilon^{2+\frac{\gamma^2}{2}}$. However, since X is independent from h one can prove a lower bound for the area of $B_\epsilon(X_\epsilon^\alpha)$ in terms of the area of $B_\epsilon(X)$ (Lemma 4.5). We will then prove an upper bound for the area of $B_\epsilon(X_\epsilon^\alpha)$ in terms of the quantities N_ϵ from Proposition 4.2, deduce from this and Lemma 4.5 an upper bound for $\text{Area}(B_\epsilon(X))$ in terms of N_ϵ and α , and optimize over α to conclude.

The following SLE estimate from [GHM15] is one of the key inputs in the proof of Proposition 4.2, and will also be used in the proof of Proposition 4.1 below.

Lemma 4.3. *Fix $u \in (0, 1)$ and for $\epsilon > 0$, let \mathcal{E}_ϵ be the event that the following is true. For each $\delta \in (0, \epsilon]$ and each $a, b \in \mathbb{R}$ with $a < b$ such that $\eta'([a, b]) \subset \mathbb{D}$ and $\text{diam}(\eta'([a, b])) \geq \delta^{1-u}$, the set $\eta'([a, b])$ contains a Euclidean ball of radius at least δ . The $\mathbb{P}[\mathcal{E}_\epsilon] = o_\epsilon^\infty(\epsilon)$.*

Proof. This is a re-statement of [GHM15, Proposition 3.1] but with a sub-polynomial bound for the rate at which $\mathbb{P}[\mathcal{E}_\epsilon] \rightarrow 0$. As explained in [GHM15, Remark 3.6], this bound follows from results in [HS16]. \square

Next we state a basic preliminary estimate for the γ -quantum area measure μ_h .

Lemma 4.4. *Suppose that h is either a circle-average embedding or a C -smooth canonical description of a γ -quantum cone for some $C \in \mathbb{R}$. For each closed set $S \subset \mathbb{D} \setminus \{0\}$ and each $\alpha > 0$,*

$$\mathbb{P}\left[\mu_h(B_\epsilon(z)) \geq \epsilon^{2+\frac{\gamma^2}{2}-\alpha}\right] \geq \epsilon^{\frac{\alpha^2}{2\gamma^2}+o_\epsilon(1)}, \quad \forall z \in S, \quad \forall \epsilon \in (0, 1) \quad (4.7)$$

at a rate depending only on C and S .

Proof. Recall that $h|_{\mathbb{D}}$ agrees in law with the restriction to \mathbb{D} of a whole-plane GFF normalized so that its circle average over $\partial\mathbb{D}$ is a constant depending only on C plus $-\gamma \log|\cdot|$ plus a random variable χ whose absolute value has a Gaussian tail. For $\epsilon > 0$ and $z \in \mathbb{C}$, let $h_\epsilon(z)$ be its circle average over $\partial B_\epsilon(z)$. For each $z \in S$ and each $\epsilon \in (0, \frac{1}{2} \text{dist}(z, \partial\mathbb{D} \cup \{0\})]$, the circle average $h_\epsilon(z) - \chi$ is Gaussian with mean $O(1)$ and variance $\log \epsilon^{-1} + O(1)$, where here $O(1)$ denotes a finite quantity which is bounded above in absolute value by constants depending only on C and S (see, e.g., the calculations in [DS11, Section 3.1]). Since $\mathbb{P}[\chi > (\log \epsilon^{-1})^2] = o_\epsilon^\infty(\epsilon)$, for $z \in S$ and $\epsilon \in (0, 1)$,

$$\mathbb{P}\left[h_\epsilon(z) \geq \frac{\alpha}{\gamma} \log \epsilon^{-1}\right] \geq \epsilon^{\frac{\alpha^2}{2\gamma^2}+o_\epsilon(1)}.$$

On the other hand, by standard tail estimates for the γ -LQG measure (see, e.g., [GHM15, Lemma 3.9]), for each $u \in (0, 1)$ it holds that

$$\mathbb{P}\left[\mu_h(B_\epsilon(z)) < \epsilon^{2+\frac{\gamma^2}{2}+u} e^{\gamma h_\epsilon(z)}\right] = o_\epsilon^\infty(\epsilon).$$

Combining the above two estimates and sending $u \rightarrow 0$ yields (4.7). \square

In the remainder of this subsection, we assume we are in the setting of Proposition 4.2. Let $\tilde{\eta}'$ be the curve η' , viewed modulo monotone parameterization, so that $\tilde{\eta}'$ is independent from h . Fix a small parameter $u \in (0, 1)$ and let

$$q := \frac{1}{1-u} > 1.$$

Let K be a deterministic upper bound for the number of times η' can hit any given point (such a bound exists; see, e.g., [GHM15, Corollary 6.5]). For $z \in \mathbb{C}$ and $i \in \{1, \dots, K\}$, let $\tau^i(z)$ be the i th time at which η' hits z (if it exists) and for $\epsilon \in (0, 1)$, define

$$\sigma_\epsilon^i(z) := \inf\{t \geq \tau^i(z) : \eta'(t) \notin B_{\epsilon q^2}(z)\}. \quad (4.8)$$

We note that the times $\tau^i(z)$ and $\sigma_\epsilon^i(z)$ depend on the parameterization of η' , hence on h , but the sets $\eta'([\tau^i(z), \sigma_\epsilon^i(z)])$ depend only on $\hat{\eta}'$, so are independent from h .

For $\alpha > 0$ and $\epsilon \in (0, 1)$, define

$$G_\epsilon^\alpha(z) := \left\{ \exists i \in \{1, \dots, K\} \text{ with } \mu_h(\eta'([\tau^i(z), \sigma_\epsilon^i(z)])) \geq \epsilon^{2+\frac{\gamma^2}{2}-\alpha} \right\} \quad (4.9)$$

and

$$X_\epsilon^\alpha := \{z \in X : G_\epsilon^\alpha(z) \text{ occurs}\}. \quad (4.10)$$

Lemma 4.5. For $\epsilon \in (0, 1)$, define the event \mathcal{E}_{ϵ^u} as in Lemma 4.3 and let X_ϵ^α be as in (4.10). For $\alpha > 0$ and $\epsilon \in (0, 1)$,

$$\mathbb{E}[\text{Area}(B_{\epsilon q^2}(X_\epsilon^\alpha)) \mid \hat{\eta}', X] \mathbb{1}_{\mathcal{E}_{\epsilon^u}} \geq \epsilon^{\frac{\alpha^2}{2\gamma^2} + o_\epsilon(1)} \text{Area}(B_{\epsilon q^2}(X)) \mathbb{1}_{\mathcal{E}_{\epsilon^u}}. \quad (4.11)$$

where the rate of the $o_\epsilon(1)$ is deterministic and depends only on S, C, u, α , and γ .

Proof. We have

$$\mathbb{E}[\text{Area}(B_{\epsilon q^2}(X_\epsilon^\alpha)) \mid \hat{\eta}', X] \mathbb{1}_{\mathcal{E}_{\epsilon^u}} = \mathbb{1}_{\mathcal{E}_{\epsilon^u}} \int_{B_{\epsilon q^2}(X)} \mathbb{P}[\text{dist}(w, X_\epsilon^\alpha) \leq \epsilon^{q^2} \mid \hat{\eta}', X] dw. \quad (4.12)$$

By definition of \mathcal{E}_{ϵ^u} , on this event for each $z \in S$ the set $\hat{\eta}'([\tau^i(z), \sigma_\epsilon^i(z)])$ contains a ball of radius at least ϵ^q for each $i \in \{1, \dots, K\}$. Since $X \subset S$ a.s. and h is independent from $(\hat{\eta}', X)$, we can apply Lemma 4.4 to get

$$\mathbb{P}[G_\epsilon^\alpha(z) \mid \hat{\eta}', X] \mathbb{1}_{\mathcal{E}_{\epsilon^u}} \geq \epsilon^{\frac{\alpha^2}{2\gamma^2} + o_\epsilon(1)} \mathbb{1}_{\mathcal{E}_{\epsilon^u}},$$

with the $o_\epsilon(1)$ satisfying the conditions in the lemma statement. For each $w \in B_{\epsilon q^2}(X)$, there is a $z \in X$ with $|z - w| \leq \epsilon^{q^2}$. For this choice of z ,

$$\mathbb{P}[\text{dist}(w, X_\epsilon^\alpha) \leq \epsilon^{q^2} \mid \hat{\eta}', X] \mathbb{1}_{\mathcal{E}_{\epsilon^u}} \geq \mathbb{P}[G_\epsilon^\alpha(z) \mid \hat{\eta}', X] \mathbb{1}_{\mathcal{E}_{\epsilon^u}} \geq \epsilon^{\frac{\alpha^2}{2\gamma^2} + o_\epsilon(1)} \mathbb{1}_{\mathcal{E}_{\epsilon^u}}. \quad (4.13)$$

By combining (4.12) and (4.13), we get (4.11). \square

Proof of Proposition 4.2. Let $u \in (0, 1)$ and let $q = (1 - u)^{-1}$ be as above. Also fix $\alpha > 0$. Define the event \mathcal{E}_{ϵ^u} as in Lemma 4.3 and the set X_ϵ^α as in (4.10). Also let

$$\hat{X}_\epsilon^\alpha := \hat{X} \cap (\eta')^{-1}(X_\epsilon^\alpha)$$

so that $X_\epsilon^\alpha \subset \eta'(\hat{X})$. We will prove an upper bound for $\mathbb{E}[\text{Area}(B_{\epsilon q^2}(X_\epsilon^\alpha)) \mathbb{1}_{\mathcal{E}_{\epsilon^u}}]$ in terms of $\mathbb{E}[\text{len}(\hat{X}_\epsilon^\alpha)]$, transfer this to an upper bound for $\mathbb{E}[\text{Area}(B_{\epsilon q^2}(X))]$ using Lemma 4.5, then optimize over all possible values of α .

Throughout the proof, for $\epsilon \in (0, 1)$ we define

$$\tilde{\epsilon} := \epsilon^{2+\frac{\gamma^2}{2}-\alpha}$$

and we recall that $N_{\tilde{\epsilon}/2}$ is the minimal number of intervals of length $\tilde{\epsilon}/2$ needed to cover \hat{X} .

Let \mathcal{I}_ϵ be a set of at most $N_{\tilde{\epsilon}/2}$ closed intervals of length $\tilde{\epsilon}/2$ whose union contains \hat{X} . Let $\mathcal{I}_\epsilon^\alpha$ be the set of those intervals $I \in \mathcal{I}_\epsilon$ such that the following is true. There is a $z \in S$ and an $i \in [1, K]_{\mathbb{Z}}$ such that $\mu_h(\eta'([\tau^i(z), \sigma_\epsilon^i(z)])) \geq \tilde{\epsilon}$ and $\tau^i(z) \in I$, where here we define $\tau^i(z)$ and $\sigma_\epsilon^i(z)$ as in (4.8) and the discussion just above with respect to the quantum mass parameterization (i.e., the parameterization of η').

For $I \in \mathcal{I}_\epsilon^\alpha$, let t_I be the infimum of the times $\tau^i(z)$ for $i \in [1, K]_\mathbb{Z}$ for which the above condition is satisfied and let $\tilde{I} := I \cap [t_I, \infty)$ be the sub-interval of I lying to the right of t_I . Also let

$$\tilde{\mathcal{I}}_\epsilon^\alpha := \left\{ \tilde{I} : I \in \mathcal{I}_\epsilon^\alpha \right\}.$$

By definition, the event $G_\epsilon^\alpha(z)$ of (4.9) occurs for each $z \in X_\epsilon^\alpha$, so for each such z there exists $i \in [1, K]_\mathbb{Z}$ such that $\mu_h(\eta'([\tau^i(z), \sigma_\epsilon^i(z)])) \geq \tilde{\epsilon}$. Since the union of the intervals in \mathcal{I}_ϵ covers \hat{X} , we infer that

$$X_\epsilon^\alpha \subset \bigcup_{\tilde{I} \in \tilde{\mathcal{I}}_\epsilon^\alpha} \eta'(\tilde{I}).$$

We now claim that on \mathcal{E}_{ϵ^u} ,

$$\text{diam } \eta(\tilde{I}) \leq 2\epsilon^{q^2} \quad \forall \tilde{I} \in \tilde{\mathcal{I}}_\epsilon^\alpha. \quad (4.14)$$

Indeed, suppose to the contrary that \mathcal{E}_{ϵ^u} but $\text{diam } \eta(\tilde{I}) > 2\epsilon^{q^2}$ for some $\tilde{I} \in \tilde{\mathcal{I}}_\epsilon^\alpha$. By definition, if we let $z = \eta'(t_I)$ be the image of the left endpoint of \tilde{I} , then $t_I = \tau_z^i$ for some $i \in [1, K]_\mathbb{Z}$ and $\mu_h(\eta([\tau_z^i, \sigma_z^{i, \epsilon}])) \geq \epsilon^{2+\frac{\gamma^2}{2}-\alpha}$ for this i . Since we are assuming that $\text{diam } \eta(\tilde{I}) > 2\epsilon^{q^2}$, by the definition (4.8) of $\sigma_\epsilon^i(z)$ we must also have $\sigma_\epsilon^i(z) \in \tilde{I}$. Therefore

$$\text{len}(\tilde{I}) = \mu_h(\eta(\tilde{I})) \geq \mu_h(\eta([\tau_z^i, \sigma_z^{i, \epsilon}])) \geq \tilde{\epsilon}.$$

But, $\text{len}(\tilde{I}) \leq \text{len}(I) = \tilde{\epsilon}/2$ by definition, so this is a contradiction.

By (4.14), on \mathcal{E}_{ϵ^u} the collection $\{\eta'(\tilde{I}) : I \in \tilde{\mathcal{I}}_\epsilon^\alpha\}$ is a covering of X_ϵ^α by at most $N_{\tilde{\epsilon}/2}$ sets of diameter at most $2\epsilon^{q^2}$. The ϵ^{q^2} -neighborhoods of these sets cover $B_{\epsilon^{q^2}}(X_\epsilon^\alpha)$ and the Euclidean area of each such neighborhood is at most a universal constant times ϵ^{2q^2} . Therefore,

$$\text{Area}(B_{\epsilon^{q^2}}(X_\epsilon^\alpha)) \mathbb{1}_{\mathcal{E}_{\epsilon^u}} \preceq \epsilon^{2q^2} N_{\tilde{\epsilon}/2} \mathbb{1}_{\mathcal{E}_{\epsilon^u}}$$

with universal implicit constant. Since \mathcal{E}_{ϵ^u} depends only on the unparameterized curve $\hat{\eta}'$, we can take expectations of both sides conditional on $\hat{\eta}', X$ to get

$$\mathbb{E}[\text{Area}(B_{\epsilon^{q^2}}(X_\epsilon^\alpha)) \mid \hat{\eta}', X] \mathbb{1}_{\mathcal{E}_{\epsilon^u}} \leq \epsilon^{2q^2} \mathbb{E}[N_{\tilde{\epsilon}/2} \mid \hat{\eta}', X] \mathbb{1}_{\mathcal{E}_{\epsilon^u}}.$$

By combining this with Lemma 4.5, we get

$$\text{Area}(B_{\epsilon^{q^2}}(X)) \mathbb{1}_{\mathcal{E}_{\epsilon^u}} \preceq \epsilon^{2q^2 - \frac{\alpha^2}{2\gamma^2} + o_\epsilon(1)} \mathbb{E}[N_{\tilde{\epsilon}/2} \mid \hat{\eta}', X] \mathbb{1}_{\mathcal{E}_{\epsilon^u}}. \quad (4.15)$$

If we assume that (4.5) holds, then we can take expectations of both sides of (4.15) to get

$$\mathbb{E}[\text{Area}(B_{\epsilon^{q^2}}(X)) \mathbb{1}_{\mathcal{E}_{\epsilon^u}}] \leq \epsilon^{2q^2 - \frac{\alpha^2}{2\gamma^2} - (2 + \frac{\gamma^2}{2} - \alpha)\beta + o_\epsilon(1)}.$$

By Lemma 4.3, $\mathbb{P}[\mathcal{E}_{\epsilon^u}] = 1 - o_\epsilon^\infty(\epsilon)$ so since $X \subset S \subset \mathbb{D}$,

$$\mathbb{E}[\text{Area}(B_{\epsilon^{q^2}}(X))] \leq \epsilon^{2q^2 - \frac{\alpha^2}{2\gamma^2} - (2 + \frac{\gamma^2}{2} - \alpha)\beta + o_\epsilon(1)} + o_\epsilon^\infty(\epsilon). \quad (4.16)$$

The exponent $2 - \frac{\alpha^2}{2\gamma^2} - (2 + \frac{\gamma^2}{2} - \alpha)\beta$ is minimized over all possible choices of α by taking $\alpha = \gamma^2\beta$. Choosing this α and sending $u \rightarrow 0$ (and hence $q \rightarrow 1$) yields (4.6). \square

4.2 Expected area of the ϵ -neighborhood of the SLE curve

The diameter estimate in Proposition 4.1 will turn out to be a straightforward consequence of the following area estimate, which we will prove using Proposition 4.2.

Proposition 4.6. *Suppose we are in the setting of Proposition 4.1. Then as $\epsilon \rightarrow 0$,*

$$\mathbb{E}[\text{Area}(B_\epsilon(\eta_{a,b} \cap S))] \leq \epsilon^{1 - \frac{\kappa'}{8} + o_\epsilon(1)} \quad (4.17)$$

at a rate depending on a, b, C, S , and κ' .

Let Z be the peanosphere Brownian motion for (h, η') , so that Z is a two-dimensional correlated Brownian motion with variances and covariances as in (1.2). Fix $a, b \in \mathbb{R}$ and let \mathcal{A} be the set of times $t \in [a, b]$ such that t is not contained in any $\pi/2$ -cone interval for Z (Definition 2.2) which is itself contained in $[a, b]$. Recall from Lemma 3.1 that $\eta'(\mathcal{A}) = \eta_{a,b}$. By Proposition 4.2, Proposition 4.6 will follow immediately from the following Brownian motion estimate.

Proposition 4.7. *Let \mathcal{A} be as above and for $\epsilon > 0$, let N_ϵ be the minimal number of intervals of length ϵ needed to cover \mathcal{A} . Then $\mathbb{E}[N_\epsilon] \preceq \epsilon^{-\kappa'/8}$.*

Proposition 4.7 will follow from a straightforward Brownian motion calculation, which we carry out in the remainder of this subsection. Throughout, we let $Z = (L, R)$ be a two-dimensional correlated Brownian motion with variances and covariances as in (1.2), as above. The following lemma is the main estimate needed for the proof of Proposition 4.7.

Lemma 4.8. *Let $a < b$ and for $\epsilon \in (0, 1)$ and $t \in [a, b - \epsilon]$ let $E_\epsilon(t)$ be the event that there does not exist a $\pi/2$ -cone time s for Z with $[t, t + \epsilon] \subset [v_Z(s), s] \subset [a, b]$. Then*

$$\mathbb{P}[E_\epsilon(t)] \preceq \epsilon^{1-\kappa'/8}((t-a) \wedge (b-t))^{-\kappa'/8} \quad (4.18)$$

with the implicit constant depending only on a and b .

Proof. Let $\underline{\tau}_\epsilon(t)$ (resp. $\bar{\tau}_\epsilon(t)$) be the smallest time $s \geq t + \epsilon$ (resp. the largest time $s \leq t + \epsilon$) at which the coordinates L and R of Z attain a simultaneous running infimum relative to time t . The time $\bar{\tau}_\epsilon(t)$ is a $\pi/2$ -cone time for Z , so if $E_\epsilon(t)$ occurs then either

$$\bar{\tau}_\epsilon(t) \geq b \quad \text{or} \quad v_Z(\bar{\tau}_\epsilon(t)) \leq a. \quad (4.19)$$

By Lemma 3.5, the conditional law of $\bar{\tau}_\epsilon(t) - \underline{\tau}_\epsilon(t)$ given $\underline{\tau}_\epsilon(t)$ is that of a single jump of a $1 - \kappa'/8$ -stable subordinator conditioned to have length at least $t + \epsilon - \underline{\tau}_\epsilon(t)$. The jumps of such a subordinator are distributed as a Poisson point process with intensity measure $cx^{-(2-\kappa'/8)} dx$ for $c > 0$ a constant depending only on κ' . Therefore,

$$\mathbb{P}[\bar{\tau}_\epsilon(t) \geq T \mid \underline{\tau}_\epsilon(t)] = \frac{(T - \underline{\tau}_\epsilon(t))^{-(1-\kappa'/8)}}{(t + \epsilon - \underline{\tau}_\epsilon(t))^{-(1-\kappa'/8)}}.$$

For fixed values of b, t , and ϵ , this last quantity is decreasing in $\underline{\tau}_\epsilon(t) \in [t, t + \epsilon]$ so is maximized at $\underline{\tau}_\epsilon(t) = t$. Averaging over all possible values of $\underline{\tau}_\epsilon(t)$ gives

$$\mathbb{P}[\bar{\tau}_\epsilon(t) \geq T] \leq \epsilon^{1-\kappa'/8}(T - t)^{-(1-\kappa'/8)}. \quad (4.20)$$

In particular, the probability of the first possibility in (4.19) satisfies

$$\mathbb{P}[\bar{\tau}_\epsilon(t) \geq b] \leq \epsilon^{1-\kappa'/8}(b - t)^{-(1-\kappa'/8)}. \quad (4.21)$$

If $v_Z(\bar{\tau}_\epsilon(t)) \leq a$, then

$$\inf_{s \in [a, t]} (L_s - L_t) \geq (L_{\bar{\tau}_\epsilon(t)} - L_t) \quad \text{and} \quad \inf_{s \in [a, t]} (R_s - R_t) \geq (R_{\bar{\tau}_\epsilon(t)} - R_t). \quad (4.22)$$

Since $\bar{\tau}_\epsilon(t)$ depends only on $(Z - Z_t)|_{[t, \infty)}$, the quantity $Z_{\bar{\tau}_\epsilon(t)} - Z_t$ is independent from $(Z - Z_t)|_{(\infty, t]}$. It follows from the estimate [Shi85, Equation (4.3)] for the probability that a Brownian motion stays in a cone for an interval of time together with a linear change of coordinates (c.f. [GHS16a, Lemma 2.7]) that the conditional probability given $(Z - Z_t)|_{[t, \infty)}$ that (4.22) holds is at most a κ' -dependent constant times

$$((L_t - L_{\bar{\tau}_\epsilon(t)}) \wedge (R_t - R_{\bar{\tau}_\epsilon(t)})) \vee ((L_t - L_{\bar{\tau}_\epsilon(t)}) \vee (R_t - R_{\bar{\tau}_\epsilon(t)}))^{\kappa'/4-1} (t - a)^{-\kappa'/8}, \quad (4.23)$$

which in turn is at most $|Z_t - Z_{\bar{\tau}_\epsilon(t)}|^{\kappa'/4} (t - a)^{-\kappa'/8}$.

To estimate the expectation of (4.23), we introduce a truncation. Let

$$G_\epsilon(t) := \left\{ |Z_{t+s} - Z_t| \leq s^{1/2} \log s^{-1}, \quad \forall s \in [t + \epsilon, b] \right\}. \quad (4.24)$$

Then $\mathbb{P}[G_\epsilon(t)] = 1 - o_\epsilon^\infty(\epsilon)$, at a rate depending only on a and b . If $G_\epsilon(t)$ occurs, then the right side of (4.23) is at most

$$(\bar{\tau}_\epsilon(t) - t)^{\kappa'/8} \log(\bar{\tau}_\epsilon(t) - t)^{-1} (t - a)^{-\kappa'/8}.$$

By integrating the estimate in (4.20), we get

$$\begin{aligned} \mathbb{P}[v_Z(\bar{\tau}_\epsilon(t)) \leq a, \bar{\tau}_\epsilon(t) \leq b, G_\epsilon(t)] &\preceq (t - a)^{-\kappa'/8} \mathbb{E} \left[(\bar{\tau}_\epsilon(t) - t)^{\kappa'/8} \log(\bar{\tau}_\epsilon(t) - t)^{-1} \mathbb{1}_{(\bar{\tau}_\epsilon(t) \leq b)} \right] \\ &\preceq \epsilon^{1-\kappa'/8} (t - a)^{-\kappa'/8}. \end{aligned} \quad (4.25)$$

We now obtain (4.18) by combining the estimates (4.21) and (4.25) and using that $\mathbb{P}[G_\epsilon(t)] = o_\epsilon^\infty(\epsilon)$. \square

Proof of Proposition 4.6. For $\epsilon > 0$ and $k \in [0, \lceil \epsilon^{-1}(b-a) \rceil]_{\mathbb{Z}}$, let $t_\epsilon^k := a + \frac{\epsilon}{k}(b-a)$. If we define the events $E_\epsilon(t)$ as in Lemma 4.8, then \mathcal{A} is contained in the union of the intervals $[t_\epsilon^k, t_\epsilon^{k+1}]$ for $k \in [0, \lceil \epsilon^{-1}(b-a) \rceil - 1]_{\mathbb{Z}}$ such that $E_\epsilon(t)$ occurs. Using the estimate of Lemma 4.8, we find that

$$\mathbb{E}[N_\epsilon] \leq \sum_{k=0}^{\lceil \epsilon^{-1}(b-a) \rceil - 1} \mathbb{P}[E_\epsilon(t_\epsilon^k)] \preceq 2 + \epsilon^{1-\kappa'/8} \sum_{k=1}^{\lceil \epsilon^{-1}(b-a) \rceil - 2} ((t_\epsilon^k - a) \wedge (b - t_\epsilon^k))^{-\kappa'/8} \preceq \epsilon^{-\kappa'/8}. \quad \square$$

4.3 Proof of Proposition 4.1

To deduce Proposition 4.1 from Proposition 4.6, we will require the following lemma.

Lemma 4.9. *Suppose we are in the setting of Proposition 4.1. For $n \in \mathbb{N}$ and $p > 2\gamma^2$,*

$$\mathbb{P} \left[\sup_{k \in [1, n]_{\mathbb{Z}}} \text{diam}(\eta'([t_{n,k-1}, t_{n,k}])) \mathbb{1}_{G_{n,k}} > n^{-(2+\gamma^2/2+p)^{-1}} \right] \leq n^{-\frac{p^2-2\gamma^2}{\gamma^2(4+\gamma^2+2p)} + o_n(1)} \quad (4.26)$$

at a rate depending on a, b, p , and γ .

Proof. Fix $u \in (0, 1)$ (which we will eventually send to 0) and let

$$\epsilon := ((b-a)n^{-1})^{(2+\gamma^2/2+p)^{-1}}.$$

Let $\mathcal{E}_{\epsilon^{\frac{1}{1-u}}}$ be the event of Lemma 4.3 with $\epsilon^{\frac{1}{1-u}}$ in place of ϵ .

By standard estimates for the LQG area measure (see, e.g., [GHS16a, Lemma 3.2] or the proof of Lemma 4.4) and recalling the relationship between h and a whole-plane GFF, for each $z \in B_{1-r}(0) \setminus B_r(0)$,

$$\mathbb{P} \left[\mu_h(B_\epsilon(z)) < \epsilon^{2+\gamma^2/2+p} \right] \leq \epsilon^{\frac{p^2}{2\gamma^2} + o_\epsilon(1)}.$$

If $\mathcal{E}_{\epsilon^{\frac{1}{1-u}}}$ occurs, then each segment of η' contained in S with diameter at least ϵ contains a Euclidean ball of radius at least $\epsilon^{\frac{1}{1-u}}$. By the union bound, the conditional probability given η' that any such segment has quantum mass smaller than $\epsilon^{2+\gamma^2/2+p}$ is at most $\epsilon^{\frac{p^2}{2\gamma^2} - 1 + o_u(1) + o_\epsilon(1)}$. Recalling the definition of ϵ , we see that the probability that there is a segment of η' contained in S with quantum mass at most $(b-a)n^{-1}$ and diameter at least $n^{-(2+\gamma^2/2+p)^{-1}}$ is at most

$$n^{-(2+\gamma^2/2+p)^{-1} \left(\frac{p^2}{2\gamma^2} - 1 \right) + o_u(1) + o_n(1)}.$$

Sending $u \rightarrow 0$ now yields the statement of the lemma. \square

Proof of Proposition 4.1. The idea of the proof is to reduce the problem of estimating squared diameters of our space-filling SLE segments to the problem of estimating the area of a small neighborhood of $\eta_{a,b} \cap S$, which we can bound using Proposition 4.6.

We start by introducing some regularity events which occur with high probability. Fix $p > 2\gamma^2$ (to be chosen later) and for $n \in \mathbb{N}$, let

$$\epsilon_n := n^{-(2+\gamma^2/2+p)^{-1}} \quad \text{and} \quad \mathcal{G}_n := \left\{ \sup_{k \in [1, n]_{\mathbb{Z}}} \text{diam}(\eta'([t_{n,k-1}, t_{n,k}])) \mathbb{1}_{G_{n,k}} \leq \epsilon_n \right\}.$$

By Lemma 4.9,

$$\mathbb{P}[\mathcal{G}_n^c] \leq n^{-\frac{p^2-2\gamma^2}{\gamma^2(4+\gamma^2+2p)}+o_n(1)}. \quad (4.27)$$

Let $u \in (0, 1/2)$ be a small constant. Let $\mathcal{E}_{\epsilon_n^u}$ be as in Lemma 4.3 with this choice of u and with $\epsilon = \epsilon_n^u$. By Lemma 4.3, $\mathbb{P}[\mathcal{E}_{\epsilon_n^u}^c] = o_n^\infty(n)$.

If \mathcal{G}_n occurs then each of the segments $\eta'([t_{n,k-1}, t_{n,k}]))$ for $k \in [1, n]_{\mathbb{Z}}$ such that $G_{n,k}$ occurs has diameter at most $\epsilon_n < \epsilon_n^u$. Hence, if also $\mathcal{E}_{\epsilon_n^u}$ occurs, then each such segment contains a Euclidean ball of radius at least $\text{diam}(\eta'([t_{n,k-1}, t_{n,k}]))^{\frac{1}{1-u}}$. Therefore,

$$\text{diam}(\eta'([t_{n,k-1}, t_{n,k}]))^2 \mathbb{1}_{G_{n,k}} \preceq \text{Area}(\eta'([t_{n,k-1}, t_{n,k}]))^{1-u} \mathbb{1}_{G_{n,k}} \quad (4.28)$$

with universal implicit constant.

On the event \mathcal{G}_n , each of the segments $\eta'([t_{n,k-1}, t_{n,k}]))$ for $k \in [1, n]_{\mathbb{Z}}$ such that $G_{n,k}$ occurs is contained in the ϵ_n -neighborhood of $\eta_{a,b} \cap S$. By combining this with (4.28), applying Hölder's inequality, and noting that the space-filling SLE segments $\eta'([t_{n,k-1}, t_{n,k}]))$ intersect only along their boundaries, we find that on $\mathcal{G}_n \cap \mathcal{E}_{\epsilon_n^u}$,

$$\begin{aligned} \sum_{k=1}^n \text{diam}(\eta'([t_{n,k-1}, t_{n,k}]))^2 \mathbb{1}_{G_{n,k}} &\preceq \sum_{k=1}^n \text{Area}(\eta'([t_{n,k-1}, t_{n,k}]))^{1-u} \mathbb{1}_{G_{n,k}} \\ &\preceq n^u \left(\sum_{k=1}^n \text{Area}(\eta'([t_{n,k-1}, t_{n,k}])) \mathbb{1}_{G_{n,k}} \right)^{1-u} \\ &\preceq n^u \text{Area}(B_{\epsilon_n}(\eta_{a,b} \cap S))^{1-u}. \end{aligned}$$

It is clear that the sum on the left is always at most $4n$, so by (4.27),

$$\mathbb{E} \left[\sum_{k=1}^n \text{diam}(\eta'([t_{n,k-1}, t_{n,k}]))^2 \mathbb{1}_{G_{n,k}} \right] \preceq n^u \mathbb{E}[\text{Area}(B_{\epsilon_n}(\eta_{a,b} \cap S))]^{1-u} + n^{1-\frac{p^2-2\gamma^2}{\gamma^2(4+\gamma^2+2p)}+o_n(1)}.$$

By Proposition 4.6, the right side of this inequality is at most

$$n^{-(1-u)(2+\gamma^2/2+p)^{-1} \left(1-\frac{2}{\gamma^2}\right) + u + o_n(1)} + n^{1-\frac{p^2-2\gamma^2}{\gamma^2(4+\gamma^2+2p)}+o_n(1)}.$$

Choosing $p = \gamma^2 + \sqrt{8\gamma^2 + 2\gamma^4 - 4}$ (so that the two powers of n are equal, modulo $o_u(1)$ and $o_n(1)$ errors) and sending $u \rightarrow 0$ yields the statement of the proposition for α as in (4.4). \square

4.4 Stability event

Recall the definition of a C -smooth canonical description of a γ -quantum cone from the beginning of this section. In this subsection we use this embedding to define a regularity event on which the diameters of space-filling SLE increments satisfy a certain bound. Suppose $\mathcal{C} = (\mathbb{C}, h, 0, \infty)$ is a γ -quantum cone and η' is a whole-plane space-filling $\text{SLE}_{\kappa'}$ independent from h and parameterized by γ -quantum mass with respect to h so that $\eta'(0) = 0$.

Following [DMS14, Section 10.4.2], for $C, t_0 \in \mathbb{R}$, a closed simple connected set $S \subset \mathbb{D} \setminus \{0\}$ with non-empty interior, $r > 0$, and $a, b \in \mathbb{R}$ with $a < b$, we let $\mathcal{A}_{C, t_0}(S, r, a, b)$ be the event that the following is true. Let $K \subset \mathbb{C} \setminus \{0\}$ be a hull (i.e., K is compact and the complement of K in the Riemann sphere is simply connected) and suppose there exists a conformal map f from the unbounded connected component of $\mathbb{C} \setminus \eta'([a, b])$ to $\mathbb{C} \setminus K$ such that the following holds.

1. $f(0) = 0$, $f(\infty) = \infty$, and $\lim_{z \rightarrow \infty} f(z)/z > 0$.
2. Suppose h' is any distribution on \mathbb{C} which agrees with the translated LQG pushforward field $h(f^{-1}(\cdot) + \eta'(t_0)) + Q \log |(f^{-1})'|$ on $\mathbb{C} \setminus K$ and whose circle average $h'_r(0)$ over $\partial B_r(0)$ is well-defined for $r \geq 1$. If we define V'_t for $t \leq 0$ as in (4.1) with h' in place of h , then (4.2) holds (for our given choice of C) with V' in place of V .

Then $K \subset S$ and $\text{diam}(K) \geq r$.

The event $\mathcal{A}_{0,0}(S, r, a, b)$ is the one considered in [DMS14, Section 10.4.2]. The event $\mathcal{A}_{C,t_0}(S, r, a, b)$ is defined in the same way but with the C -smooth canonical description in place of the 0-smooth canonical description and the addition of the translation by $\eta'(t_0)$ in the LQG coordinate change formula. As explained in [DMS14, Remark 10.15], the event $\mathcal{A}_{C,t_0}(S, r, a, b)$ is measurable with respect to the σ -algebra $\mathcal{F}_{a,b}$ generated by the field $h \circ F_{a,b}^{-1} + Q \log |(F_{a,b}^{-1})'|$, where $F_{a,b} : \mathbb{C} \setminus \overline{\mathbb{D}} \rightarrow \mathbb{C} \setminus \eta'([a, b])$ is the unique conformal transformation with $\lim_{z \rightarrow \infty} z(F_{a,b}(z) - z) = 0$. In particular, $\mathcal{A}_{C,t_0}(S, r, a, b)$ depends only on the curve-decorated quantum surface (\mathcal{C}, η'_C) , not its particular embedding into \mathbb{C} .

The reason why we include the parameters C and t_0 is so that for fixed times $a < b$, we can choose the other parameters in such a way that $\mathbb{P}[\mathcal{A}_{C,t_0}(S, r, a, b)]$ is as close to 1 as we like. In order to arrange that $\mathbb{P}[\mathcal{A}_{0,0}(S, r, a, b)]$ is close to 1, we would have to adjust a and b .

Lemma 4.10. *For each $\epsilon \in (0, 1)$ and each $a, b \in \mathbb{R}$ with $a < b$, there exists parameters C, t_0, S , and r as above such that $\mathbb{P}[\mathcal{A}_{C,t_0}(S, r, a, b)] \geq 1 - \epsilon$.*

Proof. It is proven at the end of the proof of [DMS14, Lemma 10.6] that there exists $0 < a' < b'$, a closed simply connected set $S \subset \mathbb{D} \setminus \{0\}$ with non-empty interior, and $r > 0$ such that

$$\mathbb{P}[\mathcal{A}_{0,0}(S, r, a', b') > 0] \geq 1 - \epsilon.$$

We will deduce the statement of the lemma for fixed a and b by scaling and translation. Let $C, t_0 \in \mathbb{R}$ be chosen so that

$$[a', b'] = [e^{-\gamma C} a + t_0, e^{-\gamma C} b + t_0].$$

Let $\hat{h} := h(\cdot - \eta'(t_0)) + C$ and $\hat{\eta}' := \eta'(e^{-\gamma C} \cdot + t_0) - \eta'(t_0)$. By translation invariance of the law of the pair (h, η') [DMS14, Lemma 9.3] and scale invariance of the law of the quantum cone [DMS14, Proposition 4.11], we infer that the surface-curve pairs $((\mathbb{C}, \hat{h}, 0, \infty), \hat{\eta}')$ and $((\mathbb{C}, h, 0, \infty), \eta')$ agree in law. Furthermore, $\hat{\eta}'([a, b]) = \eta'([a', b'])$.

To prove the statement of the lemma, it suffices to show that if $\mathcal{A}_{0,0}(S, r, a', b')$ occurs, then $\mathcal{A}_{C,t_0}(S, r, a, b)$ occurs with \hat{h} in place of h . Suppose K, f , and \hat{h}' are a hull, conformal map, and distribution as in the definition of $\mathcal{A}_{C,t_0}(S, r, a, b)$ with \hat{h} in place of h . Since \hat{h}' satisfies the condition (4.2) in the definition of the C -smooth canonical description and by the scaling property of the γ -LQG measure, we infer that the field $h' := \hat{h}' - C$ satisfies the condition in the definition of the 0-smooth canonical description (i.e., with $C = 0$). Since the conditions on K and f in the definition of $\mathcal{A}_{C,t_0}(S, r, a, b)$ do not depend on C and t_0 and since

$$h'|_{\mathbb{C} \setminus K} = \hat{h}(f^{-1}(\cdot) + \eta'(t_0)) + Q \log |(f^{-1})'| - C = h \circ f^{-1} + Q \log |(f^{-1})'|$$

we see that the triple (K, f, h') satisfies the conditions in the definition of $\mathcal{A}_{0,0}(S, r, a', b')$. Hence if $\mathcal{A}_{0,0}(S, r, a', b')$ occurs, then $K \subset S$ and $\text{diam}(K) \geq r$, so $\mathcal{A}_{C,t_0}(S, r, a, b)$ occurs. \square

The main reason for our interest in the events $\mathcal{A}_{C,t_0}(S, r, a, b)$ comes from the following elementary lemma, which is an easy consequence of the complex analysis facts described in [DMS14, Section 10.3].

Lemma 4.11. *Suppose $\mathcal{A}_{C,t_0}(S, r, a, b)$ occurs and let K and f be a hull and a conformal map as in the definition of $\mathcal{A}_{C,t_0}(S, r, a, b)$. Expand f as a Laurent series at ∞ ,*

$$f(z) = \alpha_{-1}z + \alpha_0 + \sum_{j=1}^{\infty} \alpha_j z^{-j}$$

for $\{\alpha_j\}_{j \geq -1}$ complex coefficients. Note that in fact α_{-1} is positive and real by our choice of f . Then $r/4 \leq \alpha_{-1} \leq 4/r$ and $|\alpha_0| \leq 4/r + 1$.

Lemma 4.11 will allow us to control how much Euclidean diameters of certain sets are distorted when we replace the quantum surface parameterized by $\eta'([a, b])$ by another quantum surface with the same area and boundary length and then embed the new quantum surface thus obtained into \mathbb{C} via the C -smooth canonical description. This is what will enable us to apply Proposition 4.1 to bound the distortions of the conformal maps in the curve-swapping argument of Section 5.

Proof of Lemma 4.11. Since the statement of the lemma does not depend on the particular choice of embedding h , we can assume without loss of generality that h is normalized so that $h(\cdot - \eta'(t_0))$ is a C -smooth centering description (so that condition 2 in the definition of $\mathcal{A}_{C,t_0}(S, r, a, b)$ holds with f equal to the identity map and K equal to the hull generated by $\eta'([a, b])$).

Let g_1 be the unique conformal map from the $\mathbb{C} \setminus \overline{\mathbb{D}}$ to the unbounded connected component of $\mathbb{C} \setminus \eta'([a, b])$ whose Laurent expansion at ∞ is given by

$$g_1(z) = \beta_{1,-1}z + \beta_{1,0} + \sum_{j=1}^{\infty} \beta_{1,j}z^{-j}$$

with $\beta_{1,-1}$ positive and real. Also let $g_2 : \mathbb{C} \setminus \overline{\mathbb{D}} \rightarrow \mathbb{C} \setminus K$ be the unique conformal map from the $\mathbb{C} \setminus \overline{\mathbb{D}}$ to the unbounded connected component of $\mathbb{C} \setminus K$ whose Laurent expansion at ∞ is given by

$$g_2(z) = \beta_{2,-1}z + \beta_{2,0} + \sum_{j=1}^{\infty} \beta_{2,j}z^{-j}.$$

Then $f = g_2 \circ g_1^{-1}$.

As $z \rightarrow \infty$, we have $g_1(z) = \beta_{1,-1}z + \beta_{1,0} + O_z(1/z)$ and pre-composing with g_1^{-1} gives $g_1^{-1}(z) = \beta_{1,-1}^{-1}z - \beta_{1,-1}^{-1}\beta_{1,0} + O_z(1/z)$ (here we use that $g_1^{-1}(z)/z$ tends to a finite constant as $z \rightarrow \infty$). Similarly, $g_2(z) = \beta_{2,-1}z + \beta_{2,0} + O_z(1/z)$. Therefore,

$$f(z) = \beta_{2,-1}\beta_{1,-1}^{-1}z - \beta_{2,-1}\beta_{1,-1}^{-1}\beta_{1,0} + \beta_{2,0} + O_z(1/z)$$

whence

$$\alpha_{-1} = \beta_{2,-1}\beta_{1,-1}^{-1} \quad \text{and} \quad \alpha_0 = -\beta_{2,-1}\beta_{1,-1}^{-1}\beta_{1,0} + \beta_{2,0}. \quad (4.29)$$

Suppose now that $\mathcal{A}_{C,t_0}(S, r, a, b)$ occurs. By standard estimates for conformal maps (see in particular [DMS14, Proposition 10.10]),

$$\frac{1}{4} \text{diam}(\eta'([a, b])) \leq \beta_{1,-1} \leq R,$$

where R is the radius of the smallest closed ball containing $\eta'([a, b])$. By definition of $\mathcal{A}_{C,t_0}(S, r, a, b)$ (applied with f equal to the identity), on this event we have $\text{diam}(\eta'([a, b])) \geq r$ and $R \leq 1$. Therefore, $\frac{1}{4}r \leq \beta_{1,-1} \leq 1$. Similarly, $\frac{1}{4}r \leq \beta_{2,-1} \leq 1$.

As explained in [DMS14, Section 10.3], $\beta_{1,0}$ is equal to $\mathbb{E}[w]$, where w is sampled according to harmonic measure from ∞ on the outer boundary of $\eta'([a, b])$ (normalized to be a probability measure). By our choice of embedding of h and the definition of $\mathcal{A}_{C,t_0}(S, r, a, b)$, we have $\eta'([a, b]) \subset S \subset \mathbb{D}$, so $|\beta_{1,0}| \leq 1$. Similarly $|\beta_{2,0}| \leq 1$.

The statement of the lemma follows by combining the above estimates with (4.29). \square

5 The curve-swapping argument

In this section we will prove Theorem 2.4. Throughout, we continue to use the notation introduced in Section 3.1.

Fix $a, b \in \mathbb{R}$ with $a < b$. Recall from Section 3.1 the surface $\mathcal{S}_{a,b}$ (resp. $\tilde{\mathcal{S}}_{a,b}$) obtained by restricting h (resp. \tilde{h}) to $\eta'([a, b])$ (resp. $\tilde{\eta}'([a, b])$) and the non-space-filling curve $\eta_{a,b,\mathcal{S}_{a,b}}$ (resp. $\tilde{\eta}_{a,b,\tilde{\mathcal{S}}_{a,b}}$) on this surface. The main input in the proof of Theorem 2.4 is the following proposition, which most of this section will be devoted to proving.

Proposition 5.1. *One has $(\tilde{\mathcal{S}}_{a,b}, \tilde{\eta}_{a,b,\tilde{\mathcal{S}}_{a,b}}) \stackrel{d}{=} (\mathcal{S}_{a,b}, \eta_{a,b,\mathcal{S}_{a,b}})$.*

Throughout most of this section we fix $a < b$ and do not make dependencies on a and b explicit.

To prove Proposition 5.1, we will start in Section 5.1 by defining for each $n \in \mathbb{N}$ and each $k \in [0, n]_{\mathbb{Z}}$ a γ -quantum cone $\mathring{\mathcal{C}}_{n,k}$ decorated by a non-space-filling curve $\mathring{\eta}_{n,k}$ and a space-filling curve $\mathring{\eta}'_{n,k}$ via a deterministic procedure involving the pairs $(\tilde{h}, \tilde{\eta}')$ and (h, η') . These curve-decorated quantum surfaces give rise to an interpolation $(\mathring{\mathcal{C}}_{n,k}, \mathring{\eta}_{n,k}, \mathring{\eta}'_{n,k})$, $k \in [0, n]_{\mathbb{Z}}$ between the laws of $(\tilde{\mathcal{S}}_{a,b}, \tilde{\eta}_{a,b}, \tilde{\mathcal{S}}_{a,b})$ and $(\mathcal{S}_{a,b}, \eta_{a,b}, \mathcal{S}_{a,b})$ in the following manner. The sub-surface of $\mathring{\mathcal{C}}_{n,0}$ (resp. $\mathring{\mathcal{C}}_{n,n}$) parameterized by $\mathring{\eta}'_{n,0}([a, b])$ (resp. $\mathring{\eta}'_{n,n}([a, b])$), decorated by the curve $\mathring{\eta}_{n,0}$ (resp. $\mathring{\eta}_{n,n}$) is exactly $(\tilde{\mathcal{S}}_{a,b}, \tilde{\eta}_{a,b}, \tilde{\mathcal{S}}_{a,b})$ (resp. has the same law as $(\mathcal{S}_{a,b}, \eta_{a,b}, \mathcal{S}_{a,b})$). Furthermore, the laws of the first and last triples $(\mathring{\mathcal{C}}_{n,0}, \mathring{\eta}_{n,0}, \mathring{\eta}'_{n,0})$ and $(\mathring{\mathcal{C}}_{n,n}, \mathring{\eta}_{n,n}, \mathring{\eta}'_{n,n})$ do not depend on n . For $n \in \mathbb{N}$ and $k \in [0, n]_{\mathbb{Z}}$ let

$$t_{n,k} := a + \frac{k}{n}(b - a). \quad (5.1)$$

Roughly speaking, each surface $\mathring{\mathcal{C}}_{n,k-1}$ is obtained from $\mathring{\mathcal{C}}_{n,k}$ by replacing the sub-surface of $\mathring{\mathcal{C}}_{n,k}$ parameterized by $\mathring{\eta}_{n,k}([t_{n,k-1}, t_{n,k}])$ by the sub-surface of our given γ -quantum cone $\tilde{\mathcal{C}}$ parameterized by $\tilde{\eta}'_{\tilde{\mathcal{C}}}([t_{n,k-1}, t_{n,k}])$.

As explained in Section 5.2, the triples $(\mathring{\mathcal{C}}_{n,k}, \mathring{\eta}_{n,k}, \mathring{\eta}'_{n,k})$ will be defined in such a way that they all have the same topology. More precisely, if we embed these curve-decorated quantum surfaces into \mathbb{C} and identify the curves with their images under these embeddings, then there exists for each $n \in \mathbb{N}$ and $k_1, k_2 \in [0, n]_{\mathbb{Z}}$ a homeomorphism $f_{n,k_1,k_2} : \mathbb{C} \rightarrow \mathbb{C}$ which satisfies $f_{n,k_1,k_2} \circ \mathring{\eta}_{n,k_1} = \mathring{\eta}_{n,k_2}$ and $f_{n,k_1,k_2} \circ \mathring{\eta}'_{n,k_1} = \mathring{\eta}'_{n,k_2}$. Furthermore, this homeomorphism can be taken to be conformal everywhere except on a small segment of the curve $\mathring{\eta}_{n,k_1}$.

The curves $\mathring{\eta}_{n,k}$ are not known to be conformally removable (they are non-space filling SLE $_{\kappa'}$ -type curves) so we cannot conclude from the above that the maps f_{n,k_1,k_2} are conformal. However, in Section 5.3 we will use Proposition 4.1 and an elementary distortion estimate for conformal maps (namely, [DMS14, Lemma 10.5]), to show that if we choose our embeddings in a certain way, then the maps $f_{n,k,k-1}$ are likely to be close to the identity when n is large. Composing these maps and taking a limit as $n \rightarrow \infty$ will show that the map $f_{n,n,0}$ (which does not depend on n) is the identity map, which will prove Proposition 5.1.

In Section 5.4, we will use Proposition 5.1 to conclude the proof of Theorem 2.4.

5.1 Defining the intermediate curves and surfaces

In this subsection we will define the curve-decorated quantum surfaces $(\mathring{\mathcal{C}}_{n,k}, \mathring{\eta}_{n,k}, \mathring{\eta}'_{n,k})$ for $n \in \mathbb{N}$ and $k \in [0, n]_{\mathbb{Z}}$ in the outline just after the statement of Proposition 5.1. This will be accomplished using the objects introduced in Section 3.1 together with the results of Section 3.5.

For $t \in [a, b]$ define the time $\tau(t) = \tau_{a,b}(t) \in [a, b]$ as in (3.3).

Recall the surfaces $\mathcal{S}_{a,b}^0 \subset \mathcal{S}_{a,b}$ and $\tilde{\mathcal{S}}_{a,b}^0 \subset \tilde{\mathcal{S}}_{a,b}$ parameterized by the bubbles cut out by $\eta_{a,b}$ and $\tilde{\eta}_{a,b}$, respectively. By Lemma 3.8, for each $k \in [0, n]_{\mathbb{Z}}$ there is a deterministic functional $F_{n,k}$ which takes in the 4-tuple

$$\left(\mathcal{S}_{-\infty,a}, \mathcal{S}_{a,\tau(t_{n,k})}^0, \eta_{a,b}, \mathcal{S}_{a,\tau(t_{n,k})}^0, \mathcal{S}_{\tau(t_{n,k}),\infty} \right)$$

and a.s. outputs the triple $(\mathcal{C}, \mathcal{S}_{a,\infty}, \eta_{a,b,c}|_{[a,t_{n,k}]})$. In particular, the functional $F_{n,k}$ imposes a conformal structure on all of $\mathcal{S}_{a,\tau(t_{n,k})}^0$ (not just on its bubbles) to get $\mathcal{S}_{a,\tau(t_{n,k})}$, conformally welds $\mathcal{S}_{a,\tau(t_{n,k})}^0$ and $\mathcal{S}_{\tau(t_{n,k}),\infty}$ together according to quantum length along their boundaries to get $\mathcal{S}_{a,\infty}$, then conformally welds $\mathcal{S}_{a,\infty}$ and $\mathcal{S}_{-\infty,a}$ together along their boundaries to get \mathcal{C} . The curve $\eta_{a,b,c}|_{[a,t_{n,k}]}$ is the image of $\eta_{a,b}, \mathcal{S}_{a,\tau(t_{n,k})}^0|_{[a,t_{n,k}]}$ under the conformal welding map.

Let $\mathring{\mathcal{S}}_{-\infty,a}$ be the $\frac{3\gamma}{2}$ -quantum wedge, independent from everything else. By Proposition 3.17 and the independence of $\mathcal{S}_{-\infty,a}$ and $(\mathcal{S}_{a,\infty}, \eta'_{\mathcal{S}_{a,\infty}})$, we have for each $k \in [1, n]_{\mathbb{Z}}$ the equality of joint laws

$$\left(\mathring{\mathcal{S}}_{-\infty,a}, \tilde{\mathcal{S}}_{a,\tau(t_{n,k})}^0, \tilde{\eta}_{a,b}, \tilde{\mathcal{S}}_{a,\tau(t_{n,k})}^0, \tilde{\mathcal{S}}_{\tau(t_{n,k}),\infty} \right) \stackrel{d}{=} \left(\mathcal{S}_{-\infty,a}, \mathcal{S}_{a,\tau(t_{n,k})}^0, \eta_{a,b}, \mathcal{S}_{a,\tau(t_{n,k})}^0, \mathcal{S}_{\tau(t_{n,k}),\infty} \right). \quad (5.2)$$

Hence we can apply $F_{n,k}$ to the left 4-tuple in (5.2) to define quantum surfaces each with a distinguished

$$\left(\mathring{C}_{n,k}, \mathring{S}_{n,k}, \mathring{\eta}_{n,k} |_{[a, t_{n,k}]}\right) := F_{n,k} \left(\mathring{S}_{-\infty, a}, \tilde{S}_{a, \tau(t_{n,k})}^0, \tilde{\eta}_{a,b}, \tilde{S}_{a, \tau(t_{n,k})}^0, \tilde{S}_{\tau(t_{n,k}), \infty}\right) \quad (5.3)$$
$$\left(\mathring{\mathcal{C}}_{n,k}, \mathring{\mathcal{S}}_{n,k}, \mathring{\eta}_{n,k}|_{[a,t_{n,k}]}\right) \stackrel{d}{=} (\mathcal{C}, \mathcal{S}_{a,\infty}, \eta_{a,b,\mathcal{C}}|_{[a,t_{n,k}]}) . \quad (5.4)$$

The figure consists of two diagrams illustrating the construction of a conformal map. The left diagram shows a domain with a central green region and a blue region, with a red boundary. A yellow region is on the left. A blue curve with arrows represents a path. The right diagram shows a similar domain with a red boundary and a blue region, with a yellow region on the left. A blue curve with arrows represents a path. The diagrams are labeled with mathematical symbols.

We will now extend the $\text{SLE}_{\kappa'}$ -type curves $\mathring{\eta}_{n,k}|_{[a,t_{n,k}]}$ to space-filling $\text{SLE}_{\kappa'}$ -type curves defined on all of $[0, \infty)$. We first extend $\mathring{\eta}_{n,k}$ to all of $[a, b]$ by concatenating it with the curve on $\mathcal{C}_{n,k}$ which corresponds to the image under the conformal welding map of the curve $\tilde{\eta}_{a,b, \tilde{\mathcal{S}}_{\tau(t_{n,k}), \infty}}|_{[t_{n,k}, \infty)}$ on $\tilde{\mathcal{S}}_{\tau(t_{n,k}), \infty}$.

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on which $\tilde{\eta}_{n,k}$ is constant. For $I = [\sigma(t), \tau(t)] \in \mathcal{I}$, define the singly marked quantum surface $\tilde{\mathcal{S}}_I := \tilde{\mathcal{S}}_{\sigma(t), \tau(t)}$ parameterized by I . Then the surfaces $\tilde{\mathcal{S}}_I$ for $I \in \mathcal{I}$ are the same as the bubbles of $\tilde{\mathcal{S}}_{a,b}^0$. Furthermore, each of the curves $\tilde{\eta}'_{\tilde{\mathcal{S}}_I}$ is a space-filling loop which starts and ends at the marked point of $\tilde{\mathcal{S}}_{\sigma(t), \tau(t)}$.

By the definitions of $\mathring{\mathcal{C}}_{n,k}$ and of $\mathring{\eta}_{n,k}$, each of the surfaces $\tilde{\mathcal{S}}_I$ for $I \in \mathcal{I}$ is a sub-surface of $\mathring{\mathcal{C}}_{n,k}$ (in fact, a sub-surface of the future wedge $\mathring{\mathcal{S}}_{n,k}$) cut out by $\mathring{\eta}_{n,k}$, the curve $\mathring{\eta}_{n,k}$ is constant on I , and the marked point of $\tilde{\mathcal{S}}_I$ corresponds to the single point $\mathring{\eta}_{n,k}(I)$. We let $\mathring{\eta}'_{n,k}|_{[a,b]}$ be the curve on $\mathring{\mathcal{C}}_{n,k}$ which agrees with the image under the conformal welding map of $\tilde{\eta}'_{\tilde{\mathcal{S}}_I}$ on each interval $I \in \mathcal{I}$ and is equal to $\mathring{\eta}_{n,k}(t)$ at each time t in the nowhere dense set $[a, b] \setminus \bigcup_{I \in \mathcal{I}} I$. Then $\mathring{\eta}'_{n,k}|_{[a,b]}$ is a continuous curve on $\mathring{\mathcal{C}}_{n,k}$. We extend $\mathring{\eta}'_{n,k}|_{[a,b]}$ to a continuous curve $\mathbb{R} \rightarrow \mathring{\mathcal{C}}_{n,k}$ which fills all of $\mathring{\mathcal{C}}_{n,k}$ as follows. For $s > b$, we define $\mathring{\eta}'_{n,k}(s)$ to be the image of $\eta'_{\mathcal{S}_{-\infty,a}}(s)$ under the conformal welding map. We also let $\mathring{\eta}'_{\mathcal{S}_{-\infty,a}}$ be a curve on $\mathring{\mathcal{S}}_{-\infty,a}$ whose conditional law given everything else is the same as the conditional law of the space-filling SLE $\eta'_{\mathcal{S}_{-\infty,a}}$ given $\mathcal{S}_{-\infty,a}$ and declare the $\mathring{\eta}'_{n,k}(s)$ is equal to the image of $\mathring{\eta}'_{\mathcal{S}_{-\infty,a}}(s)$ under the conformal welding map for $s < a$.

We record some observations about the above objects which are immediate from the construction.

- The $\frac{3\gamma}{2}$ -quantum wedge $\mathring{\mathcal{S}}_{n,k}$ is the sub-surface of $\mathring{\mathcal{C}}_{n,k}$ parameterized by $\mathring{\eta}'_{n,k}([a, \infty))$.
- The curve $\mathring{\eta}_{n,k}$ can be recovered from $\mathring{\eta}'_{n,k}$ in the same manner that $\eta_{a,b}$ is recovered from η' (i.e., by cutting out the bubbles filled in by $\mathring{\eta}'_{n,k}|_{[a,b]}$).
- Since each connected component of the interior of $\eta'([t_{n,k}, \infty))$ is contained in either $\tilde{\mathcal{S}}_{\tau(t_{n,k}), \infty}$ or in one of the bubbles $\tilde{\mathcal{S}}_I$, we see that the sub-surface of $\mathring{\mathcal{C}}_{n,k}$ parameterized by $\mathring{\eta}'_{n,k}([t_{n,k}, \infty))$ is the same as $\tilde{\mathcal{S}}_{t_{n,k}, \infty}$. Hence $\tilde{\mathcal{S}}_{t_{n,k}, \infty}$ (not just $\tilde{\mathcal{S}}_{\tau(t_{n,k}), \infty}$) is a sub-surface of $\mathring{\mathcal{C}}_{n,k}$.
- Neither $(\mathring{\mathcal{C}}_{n,n}, \mathring{\mathcal{S}}_{n,n}, \mathring{\eta}'_{n,n})$ nor $(\mathring{\mathcal{C}}_{n,0}, \mathring{\mathcal{S}}_{n,0}, \mathring{\eta}'_{n,0})$ depends on n .

The triple $(\mathring{\mathcal{C}}_{n,n}, \mathring{\mathcal{S}}_{n,n}, \mathring{\eta}'_{n,n})$ has the same law as $(\mathcal{C}, \mathcal{S}_{a,\infty}, \eta_{a,b,\mathcal{C}})$ (i.e., a γ -quantum cone decorated by an independent chordal SLE $_{\kappa'}$ segment). The surface $\mathcal{S}_{a,b}$ is a.s. equal to the union of the curve $\eta_{a,b,\mathcal{C}}$ and the set of points which it disconnects from its target point in $\mathcal{S}_{a,\infty}$. Hence $(\mathcal{S}_{a,b}, \eta_{a,b})$ has the same law as the sub-surface of $\mathring{\mathcal{S}}_{n,n}$ parameterized by the union of $\mathring{\eta}_{n,n}$ and the set of bubbles cut out by this curve in $\mathring{\mathcal{S}}_{n,k}$, equivalently the sub-surface of $\mathring{\mathcal{S}}_{n,n}$ parameterized by $\mathring{\eta}'_{n,n}([a, b])$, decorated by the curve $\mathring{\eta}_{n,n}$.

The surface $\mathring{\mathcal{C}}_{n,0}$ is obtained by conformally welding together the quantum surfaces $\mathring{\mathcal{S}}_{-\infty,a}$ and $\tilde{\mathcal{S}}_{a,\infty}$ and $\mathring{\eta}'_{n,0}$ is the concatenation of the images of $\mathring{\eta}'_{\mathcal{S}_{-\infty,a}}$ and $\tilde{\eta}'_{\tilde{\mathcal{S}}_{a,\infty}}$ under the conformal welding map. In particular, the sub-surface of $\mathring{\mathcal{S}}_{n,0}$ parameterized by $\mathring{\eta}'_{n,0}([a, b])$, decorated by the curve $\mathring{\eta}_{n,0}$, is the same as the surface $\tilde{\mathcal{S}}_{a,b}$, decorated by the curve $\tilde{\eta}'_{\tilde{\mathcal{S}}_{a,b}}$.

Therefore, in order to prove Proposition 5.1, it suffices to show that the sub-surface of $\mathring{\mathcal{S}}_{n,n}$ parameterized by $\mathring{\eta}'_{n,n}([a, b])$ and the sub-surface of $\mathring{\mathcal{S}}_{n,0}$ parameterized by $\mathring{\eta}'_{n,0}([a, b])$ a.s. agree (as quantum surfaces). This statement will follow immediately from the following proposition, which we will prove in the next subsections.

Proposition 5.2. *Almost surely, $(\mathring{\mathcal{C}}_{n,n}, \mathring{\eta}'_{n,n}) = (\mathring{\mathcal{C}}_{n,0}, \mathring{\eta}'_{n,0})$ as curve-decorated quantum surfaces.*

5.2 Embedding the intermediate surfaces

Suppose we are in the setting of Section 5.1. For $k \in [0, n]_{\mathbb{Z}}$, let $\mathring{h}_{n,k}$ be an embedding of the γ -quantum cone $\mathring{\mathcal{C}}_{n,k}$ into \mathbb{C} . The field $\mathring{h}_{n,k}$ is only defined up to a complex affine transformation, but for now we work with an arbitrary choice of embedding (we will specify a particular choice later). By a slight abuse of notation, we identify the curves $\eta_{n,k}$ and $\mathring{\eta}'_{n,k}$ with their images under this embedding, so that $\mathring{\eta}_{n,k} : [a, b] \rightarrow \mathbb{C}$ and $\mathring{\eta}'_{n,k} : \mathbb{R} \rightarrow \mathbb{C}$.

Lemma 5.3. *Fix any choice of embeddings $\mathring{h}_{n,k}$ of the quantum surfaces $\mathring{\mathcal{C}}_{n,k}$ as above. For each $k_1, k_2 \in [0, n]_{\mathbb{Z}}$, there is a homeomorphism $f = f_{n,k_1,k_2} : \mathbb{C} \rightarrow \mathbb{C}$ such that $f \circ \mathring{\eta}_{n,k_1} = \mathring{\eta}_{n,k_2}$ and $f \circ \mathring{\eta}'_{n,k_1} = \mathring{\eta}'_{n,k_2}$. Furthermore, this map is conformal on $\mathbb{C} \setminus \mathring{\eta}_{n,k_1}([t_{n,k_1 \wedge k_2}, t_{n,k_1 \vee k_2}])$ and satisfies*

$$\mathring{h}_{n,k_2}|_{\mathbb{C} \setminus \mathring{\eta}_{n,k_2}([t_{n,k_1 \wedge k_2}, t_{n,k_1 \vee k_2}])} = \mathring{h}_{n,k_1} \circ f^{-1} + Q \log |(f^{-1})'|. \quad (5.5)$$

Proof. By definition (recall the discussion just before (5.2)), the surface $\hat{\mathcal{C}}_{n,k}$ is obtained by extending the conformal structure on the beads of $\tilde{\mathcal{S}}_{a,\tau(t_{n,k})}^0$ to a conformal structure on the closure of their union, then identifying the surfaces $\hat{\mathcal{S}}_{-\infty,a}$, $\tilde{\mathcal{S}}_{a,\tau(t_{n,k})}^0$, and $\tilde{\mathcal{S}}_{\tau(t_{n,k}),\infty}$ according to quantum length along their boundaries. By condition 2 in Theorem 2.4, the equivalence relation on the boundaries of these three surfaces used to produce $\hat{\mathcal{C}}_{n,k}$ is the same as the one used to produce the original surface $\tilde{\mathcal{C}} = (\mathbb{C}, \tilde{h}, 0, \infty)$.

The curve $\hat{\eta}_{n,k}|_{[a,t_{n,k}]}$ cuts out the bubbles of the image of $\tilde{\mathcal{S}}_{a,\tau(t_{n,k})}^0$ under the conformal welding map in the same order that $\tilde{\eta}_{a,b}|_{[a,t_{n,k}]}$ cuts out the bubbles of $\tilde{\mathcal{S}}_{a,\tau(t_{n,k})}^0$ and is parameterized by the quantum mass of the bubbles it cuts out. By definition, the curve $\hat{\eta}_{n,k}|_{[t_{n,k},\infty)}$ is the image of $\tilde{\eta}_{a,b}, \tilde{\mathcal{S}}_{\tau(t_{n,k}),\infty}|_{[t_{n,k},\infty)}$ under the conformal welding map.

This holds for each $k \in [0, n]_{\mathbb{Z}}$, so for $k_1, k_2 \in [0, n]_{\mathbb{Z}}$ there exists a homeomorphism $f : \mathbb{C} \rightarrow \mathbb{C}$ which is conformal on $\mathbb{C} \setminus (\partial\hat{\eta}'_{n,k_1}([a, \infty)) \cup \hat{\eta}_{n,k_1}([a, t_{n,k_1} \vee k_2]))$ such that

$$\hat{h}_{n,k_2}|_{\mathbb{C} \setminus (\partial\hat{\eta}'_{n,k_2}([a, \infty)) \cup \hat{\eta}_{n,k_2}([a, t_{n,k_1} \vee k_2]))} = \hat{h}_{n,k_1} \circ f^{-1} + Q \log |(f^{-1})'|, \quad (5.6)$$

$f(\hat{\eta}'_{n,k_1}([a, \infty))) = \hat{\eta}'_{n,k_2}([a, \infty))$, and $f \circ \hat{\eta}_{n,k_1} = \hat{\eta}_{n,k_2}$. Since the curves $\hat{\eta}'_{n,k}$ were defined by interpolating the curves $\hat{\eta}_{n,k}$ with conformal images of segments of the same fixed curve, it follows that also $f \circ \hat{\eta}'_{n,k_1} = \hat{\eta}'_{n,k_2}$.

Now we will show that f is in fact conformal on $\mathbb{C} \setminus \hat{\eta}_{n,k_1}([t_{n,k_1} \wedge k_2, t_{n,k_1} \vee k_2])$ and check (5.5). We first argue that f extends conformally across $\partial\hat{\eta}'_{n,k_1}([a, \infty))$ using a conformal removability argument (recall Section 2.2.3). By (5.4), each of the pairs $(\hat{\mathcal{C}}_{n,k_1}, \hat{\mathcal{S}}_{n,k_1})$ agrees in law with $(\mathcal{C}, \mathcal{S}_{a,\infty})$. Since $\hat{\mathcal{S}}_{n,k_1}$ is the sub-surface of $\hat{\mathcal{C}}_{n,k_1}$ parameterized by $\hat{\eta}'_{n,k_1}([a, \infty))$, we find that $\partial\hat{\eta}'_{n,k_1}([a, \infty)) \stackrel{d}{=} \partial\eta'([a, \infty))$. By [DMS14, Footnote 9] and translation invariance of the law of η' [DMS14, Lemma 9.3], the common law of these two boundaries is that of a pair of non-crossing whole-plane $\text{SLE}_{\kappa}(2 - \kappa)$ curves, for $\kappa = 16/\kappa'$. By [DMS14, Proposition 1.8], $\partial\hat{\eta}'_{n,k_1}([a, \infty))$ is conformally removable. Hence f is a.s. conformal on $\mathbb{C} \setminus \eta_{n,k_1}([a, t_{n,k_1} \vee k_2])$.

It remains to check that f extends conformally across $\eta_{n,k_1}([a, t_{n,k_1} \wedge k_2])$. The pairs $(\tilde{\mathcal{S}}_{a,\tau(t_{n,k_1})}^0, \tilde{\eta}_{a,b}, \tilde{\mathcal{S}}_{a,\tau(t_{n,k_1})}^0)$ and $(\tilde{\mathcal{S}}_{a,\tau(t_{n,k_2})}^0, \tilde{\eta}_{a,b}, \tilde{\mathcal{S}}_{a,\tau(t_{n,k_2})}^0)$ determine the same realization of $(\mathcal{S}_{a,\tau(t_{n,k_1} \wedge k_2)}^0, \eta_{a,b}, \mathcal{S}_{a,\tau(t_{n,k_1} \wedge k_2)}^0)$. By (5.3), the sub-surfaces of $\hat{\mathcal{C}}_{n,k_1}$ and $\hat{\mathcal{C}}_{n,k_2}$, respectively, parameterized by the bubbles but out by $\hat{\eta}_{n,k_1}|_{[a,\tau(t_{n,k_1})]}$ and $\hat{\eta}_{n,k_2}|_{[a,\tau(t_{n,k_2})]}$ (i.e., the analogs of $\mathcal{S}_{a,\tau(t_{n,k_1} \wedge k_2)}^0$ for the pairs $(\hat{\mathcal{C}}_{n,k_1}, \hat{\eta}_{n,k_1})$ and $(\hat{\mathcal{C}}_{n,k_2}, \hat{\eta}_{n,k_2})$ equipped with the curves $\hat{\eta}_{n,k_1}|_{[a,\tau(t_{n,k_1})]}$ and $\hat{\eta}_{n,k_2}|_{[a,\tau(t_{n,k_2})]}$ are equivalent as curve-decorated quantum surfaces. By (5.4), the quantum surfaces $(\hat{\eta}'_{n,k_1}([a, t_{n,k_1} \wedge k_2]), \hat{h}_{n,k_1}|_{\hat{\eta}'_{n,k_1}([a, t_{n,k_1} \wedge k_2])})$ and $(\hat{\eta}'_{n,k_2}([a, t_{n,k_1} \wedge k_2]), \hat{h}_{n,k_2}|_{\hat{\eta}'_{n,k_2}([a, t_{n,k_1} \wedge k_2])})$ each have the same law as $(\mathcal{S}_{a,\tau(t_{n,k_1} \wedge k_2)}^0)$. By Lemma 3.7, these last two quantum surfaces are equivalent.

Hence there exists a homeomorphism $\hat{\eta}'_{n,k_1}([a, t_{n,k_1} \wedge k_2]) \rightarrow \hat{\eta}'_{n,k_2}([a, t_{n,k_1} \wedge k_2])$ which is conformal on the interior of $\hat{\eta}'_{n,k_1}([a, t_{n,k_1} \wedge k_2])$ and pushes forward $\hat{h}_{n,k_1}|_{\hat{\eta}'_{n,k_1}([a, t_{n,k_1} \wedge k_2])}$ to $\hat{h}_{n,k_2}|_{\hat{\eta}'_{n,k_2}([a, t_{n,k_1} \wedge k_2])}$ via the LQG coordinate change formula. By (5.6), this homeomorphism agrees with $f|_{\hat{\eta}'_{n,k_1}([a, t_{n,k_1} \wedge k_2])}$. In particular f is conformal on the interior of $\hat{\eta}'_{n,k_1}([a, t_{n,k_1} \wedge k_2])$, which contains $\hat{\eta}_{n,k_1}([a, t_{n,k_1} \wedge k_2]) \setminus \partial\hat{\eta}'_{n,k_1}([a, t_{n,k_1} \wedge k_2])$.

By (5.4), the boundary of $\hat{\eta}'_{n,k_1}([a, t_{n,k_1} \wedge k_2])$ is a locally finite union of segments of SLE_{κ} -type curves for $\kappa = 16/\kappa'$, so is conformally removable. Thus f is conformal on $\mathbb{C} \setminus \hat{\eta}'_{n,k_1}([t_{n,k_1} \wedge k_2, t_{n,k_1} \vee k_2])$ and (5.5) holds. \square

Since the curves $\hat{\eta}'_{n,k_1}$ and $\hat{\eta}'_{n,k_2}$ are space-filling, the condition that $f_{n,k_1,k_2} \circ \hat{\eta}_{n,k_1} = \hat{\eta}_{n,k_2}$ uniquely determines the map f_{n,k_1,k_2} from Lemma 5.3. This same condition also allows us to deduce that the maps f_{n,k_1,k_2} are compatible in the sense that

$$f_{n,k_1,k_3} = f_{n,k_2,k_3} \circ f_{n,k_2,k_1}^{-1}, \quad \forall k_1, k_2, k_3 \in [0, n]_{\mathbb{Z}}. \quad (5.7)$$

Recall that the embeddings $\hat{h}_{n,k}$ are only defined up to a complex affine transformation. We now specify this transformation. For $k \in [0, n]_{\mathbb{Z}}$, let $\mathbb{K}_{n,k}$ be the set of points in \mathbb{C} which are disconnected from ∞ by the image of the space-filling curve segment $\hat{\eta}'_{n,k}([a, b])$ under the embedding $\hat{h}_{n,k}$.

Each of the maps f_{n,k_1,k_2} in Lemma 5.3 restricts to a conformal map $\mathbb{C} \setminus \mathbb{K}_{n,k_1} \rightarrow \mathbb{C} \setminus \mathbb{K}_{n,k_2}$. By Laurent expansion at ∞ , we can write $f_{n,k_1,k_2} = \alpha_{n,k_1,k_2}z + \beta_{n,k_1,k_2} + O_z(1/z)$ as $z \rightarrow \infty$ for complex coefficients

α_{n,k_1,k_2} and β_{n,k_1,k_2} . By possibly applying a complex affine transformation, we can assume without loss of generality that each of the embeddings $\mathring{h}_{n,k}$ are such that $A_{n,n,k} = 1$ and $B_{n,n,k} = 0$ for each $k \in [0, n]_{\mathbb{Z}}$. By (5.7) this implies that

$$f_{n,k_1,k_2} = z + O_z(1/z) \quad \text{as } z \rightarrow \infty \quad \forall k_1, k_2 \in [0, n]_{\mathbb{Z}}. \quad (5.8)$$

There are still two complex degrees of freedom which come from pre-composing each of the fields $\mathring{h}_{n,k}$ with the same complex affine transformation. We fix all but one of these degrees of freedom (corresponding to a simultaneous rotation of all of our embeddings) by requiring that there is a conformal map $\mathbb{C} \setminus \overline{\mathbb{D}} \rightarrow \mathbb{C} \setminus \mathbb{K}_{n,n}$ which is of order $z + O_z(1/z)$ as $z \rightarrow \infty$. This is equivalent to requiring that each $\mathbb{K}_{n,k}$ has unit capacity and zero harmonic center, in the terminology of [DMS14, Section 10]. We henceforth assume that our embeddings have been selected in this manner.

Our choices of embeddings $\mathring{h}_{n,n}$ and $\mathring{h}_{n,0}$ depend only on the curve-decorated quantum surfaces $(\mathring{\mathcal{C}}_{n,n}, \mathring{\eta}'_{n,n})$ and $(\mathring{\mathcal{C}}_{n,0}, \mathring{\eta}'_{n,0})$, respectively, which we recall do not depend on n . In particular, the hulls $\mathbb{K}_{n,n}$ and $\mathbb{K}_{n,0}$ and the map $f_{n,n,0}$ do not depend on n . We emphasize this point by writing

$$\mathbb{K} = \mathbb{K}_{n,n}, \quad \widetilde{\mathbb{K}} := \mathbb{K}_{n,0}, \quad \text{and} \quad \mathbb{f} = f_{n,n,0} : \mathbb{C} \rightarrow \mathbb{C}. \quad (5.9)$$

We note that \mathbb{f} maps $\mathbb{C} \setminus \mathbb{K}$ to $\mathbb{C} \setminus \widetilde{\mathbb{K}}$ conformally. In the next subsection we will prove the following statement which immediately implies Proposition 5.2.

Proposition 5.4. *The map \mathbb{f} defined in (5.9) is a.s. equal to the identity.*

5.3 Distortion bound

Suppose we are in the setting of Section 5.2. In particular, define the embeddings $\mathring{h}_{n,k}$ of $\mathring{\mathcal{C}}_{n,k}$, the curves $\mathring{\eta}_{n,k}$ (identified with their image under this embedding), and the hulls $\mathbb{K}_{n,k}$ as in that subsection. Here we will prove Proposition 5.4, which will complete the proof of Proposition 5.1.

To lighten notation, for $k \in [1, n]_{\mathbb{Z}}$ we make the following definitions.

- Let $A_{n,k} := \mathring{\eta}_{n,k}([t_{n,k-1}, t_{n,k}])$ and $\widetilde{A}_{n,k} := \mathring{\eta}_{n,k-1}([t_{n,k-1}, t_{n,k}])$.
- Let $K_{n,k}$ (resp. $\widetilde{K}_{n,k}$) be the set of points in \mathbb{C} which are disconnected from ∞ by $A_{n,k}$ (resp. $\widetilde{A}_{n,k}$).
- Let $f_{n,k} := f_{n,k,k-1}$ and $\mathbb{f}_{n,k} := f_{n,n,k}$, so that $\mathbb{f}_{n,n} = \text{Id}$ and $\mathbb{f}_{n,0} = \mathbb{f}$.

Note that each of the hulls $K_{n,k}$ for $k \in [1, n]_{\mathbb{Z}}$ and $\widetilde{K}_{n,k+1}$ for $k \in [0, n-1]_{\mathbb{Z}}$ is contained in the hull $\mathbb{K}_{n,k}$ defined in Section 5.2. Furthermore, Lemma 5.3 implies that $f_{n,k}$ restricts to a conformal map from $\mathbb{C} \setminus A_{n,k}$ to $\mathbb{C} \setminus \widetilde{A}_{n,k}$.

For $k \in [1, n]_{\mathbb{Z}}$, let $E_{n,k}$ be the event that $\mathring{\eta}'_{n,k}([t_{n,k-1}, t_{n,k}])$ intersects $\mathring{\eta}_{n,k}$ for some (equivalently every, by Lemma 5.3) $k \in [0, n]_{\mathbb{Z}}$. On the event $E_{n,k}^c$, the curve $\mathring{\eta}_{n,k}$ is constant on $[t_{n,k-1}, t_{n,k}]$, so $A_{n,k}$ and $\widetilde{A}_{n,k}$ (hence also $K_{n,k}$ and $\widetilde{K}_{n,k}$) are singletons. By the Riemann removable singularities theorem, in this case $f_{n,k}$ is conformal on all of \mathbb{C} so by (5.8) $f_{n,k}$ is the identity map.

By (5.7),

$$\mathbb{f}_{n,k} = f_{n,k+1} \circ \cdots \circ f_{n,n} \quad \text{and} \quad \mathbb{f} = f_{n,1} \circ \cdots \circ f_{n,n}. \quad (5.10)$$

We will now bound the deviation of \mathbb{f} from the identity by bounding the deviation of each $f_{n,k}$ from the identity and summing over $k \in [1, n]_{\mathbb{Z}}$. By the discussion just above, we already know that $f_{n,k}$ is equal to the identity map on $E_{n,k}^c$ so we only need to consider the event $E_{n,k}$.

By [DMS14, Lemma 10.5] and (5.8), there are universal constants $c_1, c_2 > 0$ such that the following is true. For each $k \in [1, n]_{\mathbb{Z}}$ and each $z \in \mathbb{C}$ with $\text{dist}(z, K_{n,k}) \geq c_1 \text{diam}(K_{n,k})$, it holds that

$$|f_{n,k}(z) - z| \leq c_2 \text{diam}(K_{n,k})^2 |z - \text{hc}(K_{n,k})|^{-1} \mathbb{1}_{E_{n,k}}, \quad (5.11)$$

where here $\text{hc}(K_{n,k})$ is the harmonic center of $K_{n,k}$, i.e. the constant coefficient in the Laurent expansion of $\text{hc}(K_{n,k})$ at ∞ , and $E_{n,k}$ is the event defined just above (recall that $f_{n,k}$ is the identity on $E_{n,k}^c$, which

is why we do not need to restrict left hand side to $E_{n,k}$). Hence we are led to estimate $\text{diam}(K_{n,k})^2$ on the event $E_{n,k}$. It turns out that this is much easier to do if the embeddings $\mathring{h}_{n,k}$ are replaced by the C -smooth canonical descriptions of the surfaces $\mathcal{C}_{n,k}$ (defined in Section 4). In order to compare our given embeddings to the C -smooth canonical descriptions, we will work on the regularity event of Section 4.4. We will eventually apply Proposition 4.1 to bound the expected deviation of \mathbb{f} from the identity on this event.

Since $(\mathring{\mathcal{C}}_{n,n}, \mathring{\eta}'_{n,n}([a,b])) \stackrel{d}{=} (\mathcal{C}, \eta'([a,b]))$ has the law of a γ -quantum cone decorated by an independent space-filling $\text{SLE}_{\kappa'}$ increment, we can sample a space-filling $\text{SLE}_{\kappa'}$ curve $\mathring{\eta}''_{n,n}$ from ∞ to ∞ , independent from $\mathring{h}_{n,n}$ and parameterized by γ -quantum mass with respect to $\mathring{h}_{n,n}$, which a.s. satisfies $\mathring{\eta}''_{n,n}([a,b]) = \mathring{\eta}'_{n,n}([a,b])$ and which is conditionally independent from everything else given $(\mathring{\mathcal{C}}_{n,n}, \mathring{\eta}'_{n,n}([a,b]))$.

Let $C, t_0 \in \mathbb{R}$, let S be a closed simply connected subset of $\mathbb{D} \setminus \{0\}$ with non-empty interior, and define the event $\mathcal{A}_{C,t_0}(S, r, a, b)$ as in Section 4.4 with the curve-decorated quantum surface $((\mathring{\mathcal{C}}_{n,n} - \mathring{\eta}''_{n,n}(t_0)), \mathring{\eta}''_{n,n}) \stackrel{d}{=} (\mathcal{C}, \eta'_C)$ in place of (\mathcal{C}, η'_C) . The reason why we translate by $\mathring{\eta}''_{n,n}(t_0)$ is to cancel out the translation in condition 2 in the definition of $\mathcal{A}_{C,t_0}(S, r, a, b)$.

For $k \in [0, n]_{\mathbb{Z}}$, let $\rho_{n,k} > 0$ and $\beta_{n,k} \in \mathbb{C}$ be chosen so that the field

$$\mathring{h}_{n,k}^C := \mathring{h}_{n,k}(\rho_{n,k}^{-1}(\cdot - \beta_{n,k})) + Q \log \rho_{n,k}^{-1} \quad (5.12)$$

is a C -smooth canonical description of $\mathring{\mathcal{C}}_{n,k}$. Let

$$K_{n,k}^C := \rho_{n,k} K_{n,k} + \beta_{n,k}$$

be the embedding of $K_{n,k}$ into \mathbb{C} corresponding to $\mathring{h}_{n,k}^C$.

On $\mathcal{A}_{C,t_0}(S, r, a, b)$, we can arrange that the upper bound in (5.11) depends on $\text{diam}(K_{n,k}^C)$ (which can be bounded using Proposition 4.1) instead of $\text{diam}(K_{n,k})$.

Lemma 5.5. *On $\mathcal{A}_{C,t_0}(S, r, a, b)$, it holds for each $k \in [1, n]_{\mathbb{Z}}$ and each $z \in \mathbb{C}$ with $\text{dist}(z, K_{n,k}) \geq \frac{4c_1}{r} \text{diam}(K_{n,k}^C)$ that*

$$|f_{n,k}(z) - z| \leq \text{diam}(K_{n,k}^C)^2 \text{dist}(z, \text{CH}(K_{n,k}))^{-1} \mathbb{1}_{E_{n,k}} \mathbb{1}_{(K_{n,k}^C \subset S)} \quad (5.13)$$

with deterministic implicit constant depending only on r , where here $\text{CH}(\cdot)$ denotes the convex hull.

Proof. Recall the definition of the maps $\mathbb{f}_{n,k} = f_{n,n,k}$ from the discussion just after Proposition 5.4. By Lemma 5.3, $\mathbb{f}_{n,k}$ restricts to a conformal map $\mathbb{C} \setminus \mathbb{K} \rightarrow \mathbb{C} \setminus \mathbb{K}_{n,k}$. The restriction of the field $\mathring{h}_{n,k}^C$ to $\mathbb{C} \setminus (\rho_{n,k} \mathbb{K}_{n,k} + \beta_{n,k})$ is the LQG pushforward of the field $\mathring{h}_{n,n}|_{\mathbb{C} \setminus \mathbb{K}}$ under the conformal map $\rho_{n,k} \mathbb{f}_{n,k} + \beta_{n,k}$, which looks like $z \mapsto \rho_{n,k} z + \beta_{n,k}$ at ∞ . By the definition of $\mathcal{A}_{C,t_0}(S, r, a, b)$ (applied with $K = \rho_{n,k} \mathbb{K}_{n,k} + \beta_{n,k}$ and $f = \rho_{n,k} \mathbb{f}_{n,k} + \beta_{n,k}$), on this event

$$K_{n,k}^C \subset \rho_{n,k} \mathbb{K}_{n,k} + \beta_{n,k} \subset S. \quad (5.14)$$

Furthermore, by Lemma 4.11,

$$\frac{r}{4} \leq \rho_{n,k} \leq \frac{4}{r}. \quad (5.15)$$

Together, (5.14) and (5.15) imply that

$$\text{diam}(K_{n,k}) = \frac{\text{diam}(K_{n,k}^C)}{\rho_{n,k}} \leq \frac{4}{r} \text{diam}(K_{n,k}^C). \quad (5.16)$$

It is clear that

$$|z - \text{hc}(K_{n,k})| \geq \text{dist}(z, \text{CH}(K_{n,k})). \quad (5.17)$$

By (5.14), (5.11), (5.16), and (5.17) we find that (5.13) holds. \square

The estimate of Lemma 5.5 is our key tool for showing that the maps $f_{n,k}$ are close to the identity (recall (5.10)), but in order to apply it we need to introduce a regularity event to make sure that the inverse distance factor is not too big. To this end, fix $u > 0$ to be chosen later in a manner depending only on γ and for $n \in \mathbb{N}$ and $k \in [1, n]_{\mathbb{Z}}$, define events

$$M_{n,k}(z) := \{\text{dist}(\mathbb{f}_{n,j}(z), K_{n,j}) \geq n^{-u}, \forall j \in [k, n]_{\mathbb{Z}}\} \quad \text{and} \quad M_n(z) := M_{n,1}(z). \quad (5.18)$$

Lemma 5.6. Let $\alpha = \alpha(\kappa') > 0$ be the constant from Proposition 4.1 and suppose that $u \in (0, \alpha/4)$. With $M_{n,k}(z)$ as in (5.18) and $\mathbb{f}_{n,k}$ as in the beginning of this subsection, it is a.s. the case that on $\mathcal{A}_{C,t_0}(S, r, a, b)$,

$$\max_{k \in [1, n]_{\mathbb{Z}}} \sup\{|\mathbb{f}_{n,k-1}(z) - z| : z \in \mathbb{C}, M_{n,k}(z) \text{ occurs}\} = O_n(n^{-(\alpha/2-u)}) \quad (5.19)$$

as $n \rightarrow \infty$ along powers of 2, at a possibly random rate. In particular, on $\mathcal{A}_{C,t_0}(S, r, a, b)$, a.s.

$$\sup\{|\mathbb{f}(z) - z| : z \in \mathbb{C}, M_n(z) \text{ occurs}\} = O_n(n^{-(\alpha/2-u)}) \quad (5.20)$$

as $n \rightarrow \infty$ along powers of 2.

Proof. For $n \in \mathbb{N}$ and $k \in [1, n]_{\mathbb{Z}}$, define the following slightly modified version of $M_{n,k}(z)$:

$$\widehat{M}_{n,k}(z) := \left\{ \text{dist}(\mathbb{f}_{n,j}(z), \text{CH}(K_{n,j})) \geq \max\left\{ \frac{4c_1}{r} \text{diam}(K_{n,j}^C), \frac{1}{2}n^{-u} \right\}, \forall j \in [k, n]_{\mathbb{Z}} \right\}, \quad (5.21)$$

where here c_1 is the constant introduced just above (5.13). We will use $\widehat{M}_{n,k}(z)$ in place of $M_{n,k}(z)$ throughout most of the proof, and at the very end argue that $M_{n,k}(z) \setminus \widehat{M}_{n,k}(z)$ is unlikely when n is large.

By (5.10) and Lemma 5.5 (applied with $\mathbb{f}_{n,j}(z)$ for $j \in [k, n]_{\mathbb{Z}}$ in place of z), on $\widehat{M}_{n,k}(z) \cap \mathcal{A}_{C,t_0}(S, r, a, b)$,

$$|\mathbb{f}_{n,j-1}(z) - \mathbb{f}_{n,j}(z)| \leq n^u \text{diam}(K_{n,k}^C)^2 \mathbb{1}_{E_{n,k}} \mathbb{1}_{(K_{n,k}^C \subset S)}, \quad \forall j \in [k, n]_{\mathbb{Z}} \quad (5.22)$$

with deterministic implicit constant depending only on r . By (5.22) (applied with $\mathbb{f}_{n,j}(z)$ for $j \in [k, n]_{\mathbb{Z}}$ in place of z), for each $k \in [1, n]_{\mathbb{Z}}$,

$$\begin{aligned} |\mathbb{f}_{n,k-1}(z) - z| \mathbb{1}_{\widehat{M}_{n,k}(z)} \mathbb{1}_{\mathcal{A}_{C,t_0}(S, r, a, b)} &\leq \sum_{j=k}^n |\mathbb{f}_{n,j-1}(z) - \mathbb{f}_{n,j}(z)| \mathbb{1}_{\widehat{M}_{n,k}(z)} \mathbb{1}_{\mathcal{A}_{C,t_0}(S, r, a, b)} \\ &\leq n^u \sum_{j=1}^n \text{diam}(K_{n,j}^C)^2 \mathbb{1}_{E_{n,j}} \mathbb{1}_{(K_{n,j}^C \subset S)} \end{aligned} \quad (5.23)$$

with deterministic implicit constant depending only on r .

By (5.4), each $K_{n,j}^C$ has the same law as the set of points disconnected from ∞ by $\eta_{a,b}([t_{n,j-1}, t_{n,j}])$ if the field h is the C -smooth canonical description. We always have $\eta_{a,b}([t_{n,j-1}, t_{n,j}]) \subset \eta'([t_{n,j-1}, t_{n,j}])$ and if $\eta'([t_{n,j-1}, t_{n,j}])$ does not intersect $\eta_{a,b}$ then $\eta_{a,b}([t_{n,j-1}, t_{n,j}])$ is a single point. Hence $\text{diam}(K_{n,j}^C) \mathbb{1}_{E_{n,j}} \mathbb{1}_{(K_{n,j}^C \subset S)}$ is stochastically dominated by $\text{diam}(\eta'([t_{n,j-1}, t_{n,j}])) \mathbb{1}_{G_{n,j}}$, where

$$G_{n,j} := \{\eta'([t_{n,j-1}, t_{n,j}]) \cap \eta_{a,b} \neq \emptyset\} \cap \{\eta_{a,b}([t_{n,j-1}, t_{n,j}]) \subset S\}.$$

Hence the expectation of the right side of (5.23) satisfies

$$\mathbb{E} \left[n^u \sum_{j=1}^n \text{diam}(K_{n,j}^C)^2 \mathbb{1}_{E_{n,j}} \mathbb{1}_{(K_{n,j}^C \subset S)} \right] \leq n^u \sum_{j=1}^n \mathbb{E} [\text{diam}(\eta'([t_{n,j-1}, t_{n,j}]))^2 \mathbb{1}_{G_{n,j}}]. \quad (5.24)$$

By Proposition 4.1, the right side of this inequality is $O_n(n^{-(\alpha-u)})$, at a rate depending only on C, a, b, S , and κ' . By Chebyshev's inequality and the Borel-Cantelli lemma, it is a.s. the case that

$$\max_{j \in [1, n]_{\mathbb{Z}}} \left(\text{diam}(K_{n,j}^C)^2 \mathbb{1}_{E_{n,j}} \mathbb{1}_{(K_{n,j}^C \subset S)} \right) \leq n^u \sum_{j=1}^n \text{diam}(K_{n,j}^C)^2 \mathbb{1}_{E_{n,j}} \mathbb{1}_{(K_{n,j}^C \subset S)} = O_n(n^{-(\alpha/2-u)}) \quad (5.25)$$

as $n \rightarrow \infty$ along powers of 2.

By (5.23) and the second inequality of (5.25), if $\mathcal{A}_{C,t_0}(S, r, a, b)$ occurs then it is a.s. the case that (5.19) holds with $\widehat{M}_{n,k}(z)$ in place of $M_{n,k}(z)$.

By (5.25), it is a.s. the case that

$$\max_{j \in [1, n]_{\mathbb{Z}}} \frac{4c_1}{r} \text{diam}(K_{n,j}^C) = O_n(n^{-(\alpha/2-u)}) = o_n(n^{-u})$$

as $n \rightarrow \infty$ along powers of 2, uniformly over all $j \in [1, n]_{\mathbb{Z}}$. In particular, for large enough dyadic values of n the convex hull $\text{CH}(K_{n,j}^C)$ is contained in the $\frac{1}{2}n^{-u}$ -neighborhood of $K_{n,j}^C$. Recalling the definition (5.18) of $M_{n,k}(z)$, we see that there a.s. exists a random $n_0 \in \mathbb{N}$ such that for each dyadic $n \geq n_0$, $z \in \mathbb{C}$, and $k \in [1, n]_{\mathbb{Z}}$, the event $M_{n,k}(z) \setminus \widehat{M}_{n,k}(z)$ does not occur. Hence (5.19) follows from the analogous statement with $\widehat{M}_{n,k}(z)$ in place of $M_{n,k}(z)$.

The relation (5.20) is just (5.19) with $k = 1$. \square

To apply Lemma 5.6, we need to show that $M_n(z)$ is likely to occur for most points of \mathbb{C} when n is large. This is the purpose of the next lemma.

Lemma 5.7. *If $\mathcal{A}_{C,t_0}(S, r, a, b)$ occurs, then for each $z \in \mathbb{C} \setminus \mathring{\eta}_{n,n}([a, b])$ such that $\text{dist}(z, \mathring{\eta}_{n,n}([a, b])) \geq n^{-u/2}$ it holds that $M_n(z)$ occurs for sufficiently large $n \in \mathbb{N}$. Hence on $\mathcal{A}_{C,t_0}(S, r, a, b)$, we have $\mathbb{f}(z) = z$ for each $z \in \mathbb{C} \setminus \mathring{\eta}_{n,n}([a, b])$.*

Proof. Throughout the proof we assume that $\mathcal{A}_{C,t_0}(S, r, a, b)$ occurs. For $z \in \mathbb{C} \setminus \mathring{\eta}_{n,n}([a, b])$, let k_z^n be the largest $k \in [1, n]_{\mathbb{Z}}$ for which $\text{dist}(\mathbb{f}_{n,k}(z), \mathring{\eta}_{n,k}([a, b])) \leq \frac{1}{8}n^{-u/2}$, or $k_z^n = 0$ if no such k exists. We claim that it is a.s. the case that for each $z \in \mathbb{C}$ with $\text{dist}(z, \mathring{\eta}_{n,n}([a, b])) \geq n^{-u/2}$, we have $k_z^n = 0$ for large enough dyadic values of $n \in \mathbb{N}$.

By the definition (5.18) of $M_{n,k}(z)$, we see that if $\text{dist}(z, \mathring{\eta}_{n,n}([a, b])) \geq n^{-u/2}$, then the event $M_{n,k_z^n+1}(w)$ occurs for each large enough (deterministic) value of $n \in \mathbb{N}$ and each $w \in B_{n^{-u}}(z)$. By (5.19), it is a.s. the case that

$$\sup \left\{ |\mathbb{f}_{n,k_z^n}(w) - w| : z \in \mathbb{C}, \text{dist}(z, \mathring{\eta}_{n,n}([a, b])) \geq n^{-u/2}, w \in B_{n^{-u}}(z) \right\} = O_n(n^{-(\alpha/2-u)}), \quad (5.26)$$

as $n \rightarrow \infty$ along powers of 2.

Suppose now that $z \in \mathbb{C}$ with $\text{dist}(z, \mathring{\eta}_{n,n}([a, b])) \geq n^{-u/2}$ and $k_z^n \geq 1$. We will use (5.26) to show that $|\mathbb{f}'_{n,k_z^n}(z)|$ is close to 1 and then the Koebe quarter theorem to conclude that if n is sufficiently large, then $\text{dist}(\mathbb{f}_{n,k_z^n}(z), \mathring{\eta}_{n,k_z^n}([a, b])) \geq \frac{1}{8}n^{-u/2}$, which will contradict the definition of k_z^n .

Choose $w \in \partial B_{n^{-u}}(z)$. By Taylor's theorem with remainder,

$$\mathbb{f}_{n,k_z^n}(w) = \mathbb{f}_{n,k_z^n}(z) + \mathbb{f}'_{n,k_z^n}(z)(w - z) + O_n \left(|z - w|^2 \sup_{x \in B_{n^{-u}}(z)} |\mathbb{f}''_{n,k_z^n}(x)| \right).$$

Applying (5.26) and re-arranging gives

$$\begin{aligned} \mathbb{f}'_{n,k_z^n}(z)(z - w) &= z - w + O_n \left(|z - w|^2 \sup_{x \in B_{n^{-u}}(z)} |\mathbb{f}''_{n,k_z^n}(x)| \right) + O_n(n^{-(\alpha/2-u)}) \\ \Rightarrow \mathbb{f}'_{n,k_z^n}(z) &= 1 + O_n \left(n^{-u} \sup_{x \in B_{n^{-u}}(z)} |\mathbb{f}''_{n,k_z^n}(x)| \right) + O_n(n^{-(\alpha/2-2u)}). \end{aligned} \quad (5.27)$$

By re-scaling and applying the (easy) second coefficient case of the Bierbach-de Branges theorem, we get $|\mathbb{f}''_{n,k_z^n}(x)| \leq 4n^{u/2}|\mathbb{f}'_{n,k_z^n}(x)|$ for $x \in B_{n^{-u}}(z)$ (here we recall that \mathbb{f}_{n,k_z^n} is conformal on $B_{n^{-u/2}}(z)$). By the Koebe quarter theorem and since \mathbb{f}_{n,k_z^n} maps $\mathring{\eta}_{n,n}([a, b])$ to $\mathring{\eta}_{n,k_z^n}([a, b])$,

$$|\mathbb{f}'_{n,k_z^n}(x)| \leq 4 \frac{\text{dist}(\mathbb{f}_{n,k_z^n}(x), \mathring{\eta}_{n,k_z^n}([a, b]))}{\text{dist}(x, \mathring{\eta}_{n,k_z^n}([a, b]))}, \quad \forall x \in B_{n^{-u}}(z). \quad (5.28)$$

By definition of k_z^n , the numerator on the right is at most $\frac{1}{8}n^{-u/2} + 2n^{-u}$ and by our choice of x the denominator is at least $n^{-u/2} - 2n^{-u}$. Therefore, $|\mathbb{f}'_{n,k_z^n}(x)| = O_n(1)$ so by (5.27), $\mathbb{f}'_{n,k_z^n}(z) = 1 + o_n(1)$, at a rate which is uniform in z . In particular, (5.28) applied with $x = z$ shows that

$$\text{dist}(\mathbb{f}_{n,k_z^n}(z), \mathring{\eta}_{n,k_z^n}([a, b])) \geq \frac{1}{4}(1 + o_n(1)) \text{dist}(z, \mathring{\eta}_{n,k_z^n}([a, b])) \geq \frac{1}{4}n^{-u/2}(1 + o_n(1)).$$

This contradicts the definition of k_z^n for large enough n unless $k_z^n = 1$, so we conclude that k_z^n must be equal to 1 for large enough dyadic values of n . By (5.18) implies that $M_n(z)$ occurs for large enough dyadic values of n .

The last statement of the lemma follows from the first statement and (5.20). \square

Proof of Propositions 5.1, 5.2, and 5.4. Lemma 5.7 implies that on $\mathcal{A}_{C,t_0}(S,r,a,b)$, we have $\mathbb{f}(z) = z$ for each $z \in \mathbb{C} \setminus \dot{\eta}_{n,n}([a,b])$. Since $\dot{\eta}'_{n,n}([a,b])$ has empty interior and \mathbb{f} is continuous, a.s. \mathbb{f} is the identity map on $\mathcal{A}_{C,t_0}(S,r,a,b)$. By Lemma 4.10, for given $a < b$ we can choose C, t_0, S , and r in such a way that $\mathbb{P}[\mathcal{A}_{C,t_0}(S,r,a,b)]$ is as close to 1 as we like. We thus obtain Proposition 5.4. Propositions 5.1 and 5.2 immediately follow. \square

5.4 Proof of Theorem 2.4

Proposition 5.1 implies that for each $a, b \in \mathbb{Q}$ with $a < b$, the curve-decorated quantum surfaces $(\mathcal{S}_{a,b}, \eta_{a,b})$ and $(\tilde{\mathcal{S}}_{a,b}, \tilde{\eta}_{a,b})$ agree in law. By Lemma 3.14 (applied with $t = b$), $\mathcal{S}_{a,b}$ is a.s. determined by $(\mathcal{S}_{a,b}^0, \eta_{a,b}, \mathcal{S}_{a,b}^0)$. Hence $\tilde{\mathcal{S}}_{a,b}$ is a.s. determined by $(\tilde{\mathcal{S}}_{a,b}^0, \tilde{\eta}_{a,b}, \tilde{\mathcal{S}}_{a,b}^0)$. In particular, by (3.4), $\tilde{\mathcal{S}}_{a,b}$ is measurable with respect to the σ -algebra $\tilde{\mathcal{F}}_{a,b}$ of Theorem 2.4.

Now fix $\epsilon > 0$. By the preceding paragraph, $\tilde{\mathcal{S}}_{(j-1)\epsilon, j\epsilon} \in \tilde{\mathcal{F}}_{(j-1)\epsilon, j\epsilon} \subset \tilde{\mathcal{F}}_{(j-1)\epsilon, \infty} \cap \tilde{\mathcal{F}}_{-\infty, j\epsilon}$ for each $j \in \mathbb{Z}$. By condition 1 in the theorem statement, the quantum surfaces $\{\tilde{\mathcal{S}}_{(j-1)\epsilon, j\epsilon}\}_{j \in \mathbb{Z}}$ are independent, so

$$\{\tilde{\mathcal{S}}_{(j-1)\epsilon, j\epsilon}\}_{j \in \mathbb{Z}} \stackrel{d}{=} \{\mathcal{S}_{(j-1)\epsilon, j\epsilon}\}_{j \in \mathbb{Z}}.$$

Recall the γ -quantum cones $\mathcal{C} = (\mathbb{C}, h, 0, \infty)$ and $\tilde{\mathcal{C}} = (\mathbb{C}, \tilde{h}, 0, \infty)$ from (3.1). By condition 2 in the theorem statement, $\tilde{\mathcal{C}}$ is obtained by conformally welding the quantum surfaces $\{\tilde{\mathcal{S}}_{(j-1)\epsilon, j\epsilon}\}_{j \in \mathbb{Z}}$ in chronological order according to γ -quantum length along their boundaries, and $\tilde{\eta}'_{\tilde{\mathcal{C}}}$ fills in these surfaces in chronological order. Similar statements hold for the pair $(\mathcal{C}, \eta'_\mathcal{C})$.

Each of the surfaces $\mathcal{S}_{(j-1)\epsilon, j\epsilon}$ is parameterized by the space-filling SLE segment $\eta'_\mathcal{C}([(j-1)\epsilon, j\epsilon])$. The boundary of each of these segments is a finite union of SLE_κ -type curves for $\kappa = 16/\kappa'$. Therefore, [DMS14, Proposition 1.8] implies that a.s. the union of the boundaries of these segments is conformally removable. Hence there is only one way to conformally weld together the quantum surfaces $\mathcal{S}_{(j-1)\epsilon, j\epsilon}$ according to quantum length along their boundaries. We infer from the above descriptions of $(\tilde{\mathcal{C}}, \tilde{\eta}'_{\tilde{\mathcal{C}}})$ and $(\mathcal{C}, \eta'_\mathcal{C})$ that

$$(\tilde{\mathcal{C}}, \{\tilde{\eta}'_{\tilde{\mathcal{C}}}(j\epsilon)\}_{j \in \mathbb{Z}}) \stackrel{d}{=} (\mathcal{C}, \{\eta'_\mathcal{C}(j\epsilon)\}_{j \in \mathbb{Z}}).$$

Since $\epsilon > 0$ can be made arbitrarily small and the curves $\tilde{\eta}'_{\tilde{\mathcal{C}}}$ and $\eta'_\mathcal{C}$ are continuous, we infer that in fact $(\tilde{\mathcal{C}}, \tilde{\eta}'_{\tilde{\mathcal{C}}}) \stackrel{d}{=} (\mathcal{C}, \eta'_\mathcal{C})$.

By condition 2 in the theorem statement, Z is the peanosphere Brownian motion for each of the SLE-decorated quantum cones $(\tilde{\mathcal{C}}, \tilde{\eta}'_{\tilde{\mathcal{C}}})$ and $(\mathcal{C}, \eta'_\mathcal{C})$, so by [DMS14, Theorem 1.14], a.s. $(\tilde{\mathcal{C}}, \tilde{\eta}'_{\tilde{\mathcal{C}}}) = (\mathcal{C}, \eta'_\mathcal{C})$ as curve-decorated quantum surfaces. Hence there a.s. exists a conformal automorphism of \mathbb{C} which fixes 0 (i.e., multiplication by a complex number) which takes η' to $\tilde{\eta}'$. Since $\Phi \circ \eta' = \tilde{\eta}'$ and η' is space-filling, it follows that this conformal automorphism coincides with Φ . \square

6 Characterization of chordal SLE on a bead of a thin wedge

In this section we will state and prove an analog of Theorem 2.4 for a chordal $\text{SLE}_{\kappa'}$ on a single bead of a $\frac{3\gamma}{2}$ -quantum wedge.

Suppose h^b is an embedding into $(\mathbb{H}, 0, \infty)$ of a single bead of a $\frac{3\gamma}{2}$ -quantum wedge with specified area and/or left/right boundary lengths. Let $\eta^b : [0, \infty) \rightarrow \mathbb{H}$ be an independent chordal $\text{SLE}_{\kappa'}$ from 0 to ∞ in \mathbb{H} , with some choice of parameterization. For $t \geq 0$, let L_t^b (resp. R_t^b) be the ν_{h^b} -length of the boundary arc of the unbounded connected component of $\mathbb{H} \setminus \eta^b([0, t])$ which lies to the left (resp. right) of $\eta^b(t)$ and define the *boundary length process* $Z_t^b := (L_t^b, R_t^b)$. Note that $Z_0^b = (\mathfrak{l}_L, \mathfrak{l}_R)$.

The process L_t^b (resp. R_t^b) is a right continuous process with left hand limits with no upward jumps, but with a downward jump whenever η^b disconnects a bubble from ∞ on its left (resp. right) side. The magnitude of the downward jump corresponds to the boundary length of the bubble. In contrast to the peanosphere Brownian motion Z from Theorem 2.4, the process Z^b does not a.s. determine the pair (h^b, η^b) modulo conformal maps for any choice of parameterization because it does not encode all of the structure of the disks which are cut off by the curve. It does, however, encode the equivalence class of (\mathbb{D}, η) modulo curve-preserving homeomorphisms.

For our chordal SLE characterization theorem, we will parameterize η^b in such a way that the boundary length process Z^b does encode the quantum mass of the bubbles cut off from ∞ by η^b . In particular, we will parameterize η^b by the μ_{h^b} -mass it disconnects from ∞ , which is defined as follows.

Definition 6.1. Suppose X is a topological space equipped with a measure μ , $x \in X$, and $\eta : [0, T] \rightarrow X$ is a curve with $\eta(T) = x$. We say that η is *parameterized by the μ -mass it disconnects from x* if the following is true. For $t \in [0, T]$, let U_t be the connected component of $X \setminus \eta([0, t])$ containing x and let $K_t = X \setminus U_t$ be the hull generated by $\eta([0, t])$. Then for each $t \in [0, T]$,

$$\eta(t) = \eta(\inf\{s \in [0, T] : \mu(K_s) \geq t\}). \quad (6.1)$$

We note that whenever a curve parameterized by the μ -mass it disconnects from x cuts off some region U from x , the curve remains constant on a time interval of length $\mu(U)$ immediately following the disconnection time.

Suppose the $\text{SLE}_{\kappa'}$ curve η^b defined at the beginning of this section is parameterized by the μ_{h^b} -mass it disconnects from ∞ . Then the process Z^b encodes the quantum areas of the connected components of $\mathbb{H} \setminus \eta^b([0, t])$ in the following manner. For $t \in [0, \mathfrak{a}]$, let $[\sigma^b(t), \tau^b(t)]$ be the largest interval containing t on which Z^b is constant (note that this interval may be a single point). Also let D_t be the last bubble disconnected from ∞ by η^b before time t (or $D_t = \eta^b(t)$ if no such last bubble exists). As noted above, the boundary length of D_t is encoded by the downward jump of Z^b at time t , and this together with our choice of parameterization gives

$$\mu_{h^b}(D_t) = \tau^b(t) - \sigma^b(t) \quad \text{and} \quad \nu_{h^b}(\partial D_t) = |Z_{\sigma^b(t)} - \lim_{s \rightarrow \sigma^b(t)^-} Z_s^b|. \quad (6.2)$$

We now state our chordal $\text{SLE}_{\kappa'}$ characterization theorem.

Theorem 6.2 (Chordal SLE characterization on a bead of a thin wedge). *Let $\kappa' \in (4, 8)$ and $\gamma = 4/\sqrt{\kappa'} \in (\sqrt{2}, 2)$. Let $(\mathfrak{a}, \mathfrak{l}^L, \mathfrak{l}^R) \in (0, \infty)^3$ and suppose we are given a random triple $(\tilde{h}^b, \tilde{\eta}^b, Z^b)$ where \tilde{h}^b is an embedding into $(\mathbb{H}, 0, \infty)$ of a single bead of a $\frac{3\gamma}{2}$ -quantum wedge with area \mathfrak{a} and left/right boundary lengths \mathfrak{l}^L and \mathfrak{l}^R , $\tilde{\eta}^b : [0, \mathfrak{a}] \rightarrow \overline{\mathbb{H}}$ is a random continuous curve parameterized by the $\mu_{\tilde{h}^b}$ -mass it disconnects from ∞ (Definition 6.1), and $Z^b = (L^b, R^b)$ has the law of the boundary length process of a chordal $\text{SLE}_{\kappa'}$ on a doubly marked quantum disk with area \mathfrak{a} and left/right boundary lengths \mathfrak{l}^L and \mathfrak{l}^R , parameterized by the quantum mass it disconnects from ∞ . Assume that the following conditions are satisfied.*

1. (Law of complementary connected components) *For $t \in [0, \mathfrak{a}]$, let $\tilde{\mathcal{W}}_t^b$ be the doubly marked quantum surface obtained by restricting \tilde{h}^b to the unbounded connected component of $\mathbb{H} \setminus \tilde{\eta}^b([0, t])$, with marked points $\tilde{\eta}^b(t)$ and ∞ . If we condition on $Z^b|_{[0, t]}$ and the time $\tau^b(t)$ from (6.2), then the conditional law of $\tilde{\mathcal{W}}_t^b$ is that of a single bead of a $\frac{3\gamma}{2}$ -quantum wedge with area $\mathfrak{a} - \tau^b(t)$ and left/right boundary lengths L_t^b and R_t^b . The conditional law of the collection of singly marked quantum surfaces obtained by restricting \tilde{h}^b to the bubbles disconnected from ∞ by $\tilde{\eta}^b$ before time t , each marked by the point where $\tilde{\eta}^b$ finishes tracing its boundary, is that of a collection of independent singly marked quantum disks parameterized by the intervals of time in $[0, \tau^b(t)]$ on which Z^b is constant, with areas and boundary lengths determined by Z^b as in (6.2), and these singly marked quantum surfaces are conditionally independent from $\tilde{\mathcal{W}}_t^b$.*
2. (Topology and consistency) *The topology of $(\mathbb{H}, \tilde{\eta}^b)$ is determined by Z^b in the same manner as the topology of an $\text{SLE}_{\kappa'}$, i.e. there is a pair $((\mathbb{H}, h^b, 0, \infty), \eta^b)$ consisting of a bead of a $\frac{3\gamma}{2}$ -quantum wedge with area \mathfrak{a} and left/right boundary lengths \mathfrak{l}^L and \mathfrak{l}^R and an independent chordal $\text{SLE}_{\kappa'}$ from 0 to ∞ in \mathbb{H} parameterized by the μ_{h^b} -mass it disconnects from ∞ and a homeomorphism $\Phi^b : \mathbb{H} \rightarrow \mathbb{H}$*

with $\Phi^b \circ \eta^b = \tilde{\eta}^b$. Moreover, Φ^b pushes forward the γ -quantum length measure with respect to h^b on the boundary of the unbounded connected component of $\mathbb{H} \setminus \tilde{\eta}^b([0, t])$ to the γ -quantum length measure with respect to \tilde{h}^b on the boundary of the unbounded connected component of $\mathbb{H} \setminus \tilde{\eta}^b([0, t])$ for each $t \in [0, \mathfrak{a}] \cap \mathbb{Q}$.

Then $(\tilde{h}^b, \tilde{\eta}^b)$ is an embedding into $(\mathbb{H}, 0, \infty)$ of a single bead of a $\frac{3\gamma}{2}$ -quantum wedge together with an independent chordal $\text{SLE}_{\kappa'}$ from 0 to ∞ in \mathbb{H} parameterized by the $\mu_{\tilde{h}^b}$ -mass it disconnects from ∞ .

We will prove in Lemma 6.4 below that the hypotheses of Theorem 6.2 are satisfied when $(\tilde{h}^b, \tilde{\eta}^b) = (h^b, \eta^b)$, at least for almost every triple $(\mathfrak{a}, \mathfrak{l}^L, \mathfrak{l}^R)$. This statement is not needed for the proof of Theorem 6.2 but the theorem statement is vacuous without it.

As in Theorem 1.1, the conditions of Theorem 6.2 enable us to define the γ -quantum length measure with respect to \tilde{h}^b on the boundary of each connected component of $\mathbb{H} \setminus \tilde{\eta}^b([0, t])$ for all $t \in [0, \mathfrak{a}]$ simultaneously. In fact, by considering a time immediately before $\tilde{\eta}^b$ disconnects a given bubble from ∞ , we see that the homeomorphism Φ^b of condition 2 is boundary-length preserving on each connected component of $\mathbb{H} \setminus \eta^b([0, t])$ for each $t \in [0, \mathfrak{a}]$.

In the remainder of this section we will deduce Theorem 6.2 from Theorem 2.4. We will first prove a variant of Theorem 6.2 where the area/boundary length triple $(\mathfrak{a}, \mathfrak{l}^L, \mathfrak{l}^R)$ is random. The reason for this is that, as we will see below, this enables us to couple $(\tilde{h}^b, \tilde{\eta}^b)$ with a pair $(\tilde{h}, \tilde{\eta}')$ consisting of an embedding into $(\mathbb{C}, 0, \infty)$ of a γ -quantum cone and a space-filling curve in \mathbb{C} satisfying the hypotheses of Theorem 2.4. The quantum surface $(\mathbb{H}, \tilde{h}^b, 0, \infty)$ will be one of the beads of the future quantum surface $\tilde{\mathcal{S}}_{0, \infty}$ and the curve $\tilde{\eta}'$ restricted to this bead will be obtained from $\tilde{\eta}^b$ by filling in each of the complementary connected components of $\tilde{\eta}^b$ by a space-filling $\text{SLE}_{\kappa'}$ loop. Such a coupling does not work for a deterministic triple $(\mathfrak{a}, \mathfrak{l}^L, \mathfrak{l}^R)$ since a $\frac{3\gamma}{2}$ -quantum wedge does *not* a.s. have a bead with area \mathfrak{a} and left/right boundary lengths \mathfrak{l}^L and \mathfrak{l}^R .

We now proceed with the details. Let \mathbb{M} be the infinite measure on beads of a $\frac{3\gamma}{2}$ -quantum wedge (recall [DMS14, Definition 4.13]) and let \mathfrak{m} be the infinite measure on $(0, \infty)^3$ which is the pushforward of \mathbb{M} under the function which assigns to each bead its vector of area and left/right quantum boundary lengths. Let $\mathfrak{A} \subset (0, \infty)^3$ be a Borel set such that $\mathfrak{m}(\mathfrak{A})$ is finite and positive.

Throughout most of this section, we assume that we are in the setting of Theorem 6.2 except that the triple $(\mathfrak{a}, \mathfrak{l}^L, \mathfrak{l}^R)$ is sampled from the probability measure $\mathfrak{m}(\mathfrak{A})^{-1} \mathfrak{m}|_{\mathfrak{A}}$, instead of being deterministic. We assume that the conditions in the theorem statement all hold for the conditional law given $(\mathfrak{a}, \mathfrak{l}^L, \mathfrak{l}^R)$. We will only return to the case of deterministic $(\mathfrak{a}, \mathfrak{l}^L, \mathfrak{l}^R)$ at the very end of the proof.

Let $\mathcal{C} = (\mathbb{C}, h, 0, \infty)$ be a γ -quantum cone and let η' be a whole-plane space-filling $\text{SLE}_{\kappa'}$ independent from h and parameterized by γ -quantum mass with respect to h in such a way that $\eta'(0) = 0$. Let Z be the corresponding peanosphere Brownian motion. Define the quantum surfaces $\mathcal{S}_{a, b}$ parameterized by $\eta'([a, b])$ for $a < b$ as in (3.2). Recall in particular that each of the future beaded surfaces $\mathcal{S}_{t, \infty}$ for $t \in \mathbb{R}$ has the law of a $\frac{3\gamma}{2}$ -quantum wedge.

Let \mathcal{B} be the first bead of the $\frac{3\gamma}{2}$ -quantum wedge $\mathcal{S}_{0, \infty}$ whose vector of area and left/right quantum boundary lengths belongs to \mathfrak{A} . Let \underline{T} and \overline{T} be the times at which η' starts and finishes filling in this bead, so that

$$\mathcal{B} = \mathcal{S}_{\underline{T}, \overline{T}} = \left(\eta'([\underline{T}, \overline{T}]), h|_{\eta'([\underline{T}, \overline{T}])}, \eta'(\underline{T}), \eta'(\overline{T}) \right) \quad (6.3)$$

and $\overline{T} - \underline{T}$ is the quantum mass of \mathcal{B} . By our choice of \mathfrak{A} and since the beads of $\mathcal{S}_{0, \infty}$ are a Poisson point process sampled from \mathbb{M} , we find that \mathcal{B} is well-defined a.s. and that the law of \mathcal{B} is that of a single bead of a $\frac{3\gamma}{2}$ -quantum wedge conditioned to have area and left/right quantum boundary lengths in \mathfrak{A} , i.e. $\mathcal{B} \stackrel{d}{=} (\mathbb{H}, h^b, 0, \infty)$.

Let $\eta_{0, \infty}$ be the curve obtained from η' by skipping the bubbles filled in by η' in the time interval $[0, \infty)$, as in Section 3.1, and define the times $\sigma(t) = \sigma_{0, \infty}(t)$ and $\tau = \tau_{0, \infty}(t)$ for $t \geq 0$ as in (3.3), so that $\eta_{0, \infty}$ is constant on each time interval $[\sigma(t), \tau(t)]$.

By [DMS14, Footnote 9] the conditional law of $\eta_{0, \infty}(\cdot - \underline{T})|_{[\underline{T}, \overline{T}]}$ given \tilde{h} and $\eta'(\underline{T}, \overline{T})$ is that of a chordal $\text{SLE}_{\kappa'}$ from $\eta'(\underline{T})$ to $\eta'(\overline{T})$ in $\eta'([\underline{T}, \overline{T}])$, parameterized by the μ_h -mass of the region it disconnects from $\eta'(\overline{T})$. Therefore the curve-decorated quantum surface $(\mathcal{B}, \eta_{0, \infty, \mathcal{B}}(\cdot - \underline{T}))$ agrees in law with $((\mathbb{H}, h^b, 0, \infty), \eta^b)$ (the latter viewed as a curve-decorated quantum surface).

Thus, we can couple $(\tilde{h}^b, \tilde{\eta}^b)$ and (h^b, η^b) with (h, η') in such a way that a.s. $(\mathcal{B}, \eta_{0,\infty,\mathcal{B}}(\cdot - \underline{T}))$ and $((\mathbb{H}, h^b, 0, \infty), \eta^b)$ agree as curve-decorated quantum surfaces and (h, η') is conditionally independent from $(\tilde{h}^b, \tilde{\eta}^b)$ and (h^b, η^b) given $(\mathcal{B}, \eta_{0,\infty,\mathcal{B}}(\cdot - \underline{T}))$. Henceforth fix such a coupling.

For our choice of coupling, $\mathfrak{a} = \overline{T} - \underline{T}$ and corresponding boundary length process appearing in Theorem 6.2 is given by

$$Z^b = (L^b, R^b) = \{Z_{\tau(t) - \underline{T}} - 2Z_{\underline{T}} + Z_{\overline{T}}\}_{t \in [0, \mathfrak{a}]}.$$
 (6.4)

Furthermore, the time intervals on which Z^b is constant defined just above Theorem 6.2 satisfy $\sigma^b(t) = \sigma(t) - \underline{T}$ and $\tau^b(t) = \tau(t) - \underline{T}$.

The above coupling together with the results of Section 3 allows us to deduce the following lemma, which tells us that for random $(\mathfrak{a}, \mathfrak{l}^L, \mathfrak{l}^R)$, the conditions of Theorem 6.2 are satisfied in the special case when $(\tilde{h}^b, \tilde{\eta}^b) = (h^b, \eta^b)$.

Lemma 6.3. *For any choice of set $\mathfrak{A} \subset (0, \infty)^3$ as above, a slightly stronger version of condition 1 in Theorem 6.2 holds with (h^b, η^b) in place of $(\tilde{h}^b, \tilde{\eta}^b)$.*

More precisely, for $t \in [0, \mathfrak{a}]$ let \mathcal{W}_t^b be the doubly marked quantum surface obtained by restricting h^b to the unbounded connected component of $\mathbb{H} \setminus \eta^b([0, t])$, with marked points $\eta^b(t)$ and ∞ . If we condition on $(\mathfrak{a}, \mathfrak{l}^L, \mathfrak{l}^R)$, $Z^b|_{[0, t]}$, and the time $\tau^b(t)$ from (6.2), then the conditional law of \mathcal{W}_t^b is that of a single bead of a $\frac{3\gamma}{2}$ -quantum wedge with area $\mathfrak{a} - \tau^b(t)$ and left/right boundary lengths L_t^b and R_t^b . The conditional law of the collection of singly marked quantum surfaces obtained by restricting h^b to the bubbles disconnected from ∞ by η^b before time t , each marked by the point where η^b finishes tracing its boundary, is that of a collection of independent singly marked quantum disks parameterized by the intervals of time in $[0, \tau^b(t)]$ on which Z^b is constant, with areas and left/right quantum boundary lengths determined by Z^b as in (6.2).

Furthermore, the curve-decorated quantum surfaces $(\mathcal{W}_t^b, \eta_{\mathcal{W}_t^b}^b)$ and $(\mathcal{B} \setminus \mathcal{W}_t^b, \eta_{\mathcal{B} \setminus \mathcal{W}_t^b}^b)$ are conditionally independent given $(\mathfrak{a}, \mathfrak{l}^L, \mathfrak{l}^R)$, Z_t^b , and $\tau^b(t)$.

We note that the last statement about curve-decorated quantum surfaces is not part of the hypotheses of Theorem 6.2, and is stronger than the independence assertion in condition 1 of Theorem 6.2.

Proof of Lemma 6.3. Recall the γ -quantum cone/space-filling $\text{SLE}_{\kappa'}$ pair (h, η') defined above. For $t \geq 0$, let $P_{t,\infty}$ be the function as in (3.5) which encodes the areas and left/right boundary lengths of the beads of the surface $\mathcal{S}_{t,\infty}$. Then $[\underline{T}, \overline{T}]$ is the leftmost interval of times on which $P_{0,\infty}(s) \in \mathfrak{A}$ and $(\mathfrak{a}, \mathfrak{l}^L, \mathfrak{l}^R)$ is the value of $P_{0,\infty}$ on this interval. Furthermore, if we define the time intervals $[\sigma(s), \tau(s)] \subset [0, \infty)$ on which η' is constant as above, then for each $t \in [0, \mathfrak{a}] = [0, \overline{T} - \underline{T}]$, we have by our choice of coupling that

$$\mathcal{W}_t^b = \mathcal{S}_{\tau(\underline{T}+t), \overline{T}}.$$
 (6.5)

By (6.4), Z^b is determined by $\{Z_{\tau(s)} : s \geq 0\}$. It therefore follows from Proposition 3.9 that the conditional law given Z^b of the collection of singly marked quantum surfaces obtained by restricting h^b to the bubbles disconnected from ∞ by η^b before time t is as in the statement of the lemma.

To study the conditional law of $(\mathcal{W}_t^b, \eta_{\mathcal{W}_t^b}^b)$, let $\mathcal{G}_r = \sigma((Z - Z_r)|_{(-\infty, r]}) \vee \sigma(P_{r,\infty})$ for $r \in \mathbb{R}$ be the σ -algebra of Lemma 3.13. Our above description of the times \underline{T} and \overline{T} in terms of $P_{0,\infty}$ shows that \underline{T} , \overline{T} , and that the area and left/right quantum boundary lengths $(\mathfrak{a}, \mathfrak{l}^L, \mathfrak{l}^R)$ of \mathcal{B} are \mathcal{G}_0 -measurable. Furthermore, the time $\tau(t)$ for $t \geq 0$ is \mathcal{G}_t -measurable since $\tau(t)$ is the time when η' begins filling in the bead of $\mathcal{S}_{t,\infty}$ which it finishes filling in at the right endpoint of the largest interval containing t on which $P_{t,\infty}$ is constant.

For $n \in \mathbb{N}$, let $\underline{T}_n = 2^{-n} \lceil 2^n \underline{T} \rceil$, so that each \underline{T}_n is a $\{\mathcal{G}_r\}_{r \in \mathbb{R}}$ stopping time which takes on only countably many possible values and \underline{T}_n decreases to \underline{T} . Also let $\overline{T}_n(t)$ for $t \in (0, \overline{T} - \underline{T})$ be the first $s > \tau(\underline{T}_n + t)$ at which the Brownian motions L and R attain a simultaneous running infimum relative to time $\underline{T}_n + t$, so that $\mathcal{S}_{\tau(\underline{T}_n + t), \overline{T}_n(t)}$ is a bead of the quantum surface $\mathcal{S}_{\underline{T}_n + t, \infty}$.

For $t \in (0, \underline{T} - \overline{T})$, a.s. $t \in (\sigma(\underline{T} + t), \tau(\underline{T} + t))$. Hence a.s. $\tau(\underline{T}_n + t) = \tau(\underline{T} + t)$ for large enough $n \in \mathbb{N}$. Furthermore, since $[\sigma(t), \tau(t)]$ is a maximal $\pi/2$ -cone interval for Z in $[0, \infty)$ (Lemma 3.1), Z does not attain a running infimum relative to time $\underline{T} + t$ during the time interval $[\underline{T} + t, \overline{T}]$, so a.s. $\overline{T}_n(t) = \overline{T}$ for large enough $n \in \mathbb{N}$. Each of the times $\tau(\underline{T}_n + t)$ and $\overline{T}_n(t)$ is $\mathcal{G}_{\underline{T}_n + t}$ -measurable. Since $\mathcal{S}_{\tau(\underline{T}_n + t), \overline{T}_n(t)}$ is a bead of $\mathcal{S}_{\underline{T}_n + t, \infty}$, the conditional law of $\mathcal{S}_{\tau(\underline{T}_n + t), \overline{T}_n(t)}$ given $\mathcal{G}_{\underline{T}_n + t}$ is that of a single bead of a $\frac{3\gamma}{2}$ -quantum wedge with given area and left/right boundary lengths.

Taking a limit as $n \rightarrow \infty$, applying the reverse martingale convergence theorem, and recalling (6.5) shows that the conditional law of \mathcal{W}_t^b given \mathcal{G}_{T+t} is that of a single bead of a $\frac{3\gamma}{2}$ -quantum wedge with given area and left/right boundary lengths. It is clear from the domain Markov property that the conditional law of $\eta|_{[t,a]}$ given \mathcal{G}_{T+t} and \mathcal{W}_t^b is that of a chordal $\text{SLE}_{\kappa'}$ between the two marked points of \mathcal{W}_t^b , parameterized by the quantum mass it disconnects from its target point. In particular, the conditional law of $(\mathcal{W}_t^b, \eta_{\mathcal{W}_t^b}^b)$ depends only on the area and left/right quantum boundary lengths of \mathcal{W}_t^b , which are determined by $(\mathbf{a}, \mathbf{l}^L, \mathbf{l}^R)$, Z_t^b , and $\tau^b(t)$.

The restricted Brownian motion $(Z - Z_{T+t})|_{(-\infty, T+t]}$, the vector $(\mathbf{a}, \mathbf{l}^L, \mathbf{l}^R)$, and the time $\tau^b(t) = \tau(t) - T$ are all \mathcal{G}_{T+t} -measurable. Since each curve-decorated quantum surface $(\mathcal{S}_{a,b}, \eta'_{\mathcal{S}_{a,b}})$ is a.s. determined by $(Z - Z_a)|_{[a,b]}$ and $\eta'_{\mathcal{S}_{-\infty, T+t}}$ is a.s. determined by $(\mathcal{S}_{-\infty, T+t}, \eta'_{\mathcal{S}_{-\infty, T+t}})$ and $\tau(t)$, we infer that $(\mathcal{B} \setminus \mathcal{W}_t^b, \eta_{\mathcal{B} \setminus \mathcal{W}_t^b}^b) \in \mathcal{G}_{T+t}$. In particular, the collection of singly marked quantum surfaces obtained by restricting h^b to the bubbles disconnected from ∞ by η^b before time t is \mathcal{G}_{T+t} -measurable. The statement of the lemma follows by combining this with our above description of the conditional law of this collection of singly marked quantum surfaces given Z^b and the conditional law of \mathcal{W}_t^b given \mathcal{G}_{T+t} . \square

We record the following consequence of Lemma 6.3, which says that the hypotheses of Theorem 6.2 in the case when $(\tilde{h}^b, \tilde{\eta}^b) = (h^b, \eta^b)$ are satisfied for almost every deterministic choice of area and left/right boundary length vector $(\mathbf{a}, \mathbf{l}^L, \mathbf{l}^R)$.

Lemma 6.4. *For Lebesgue-a.e. triple $(\mathbf{a}_0, \mathbf{l}_0^L, \mathbf{l}_0^R) \in (0, \infty)^3$, the conclusion of Lemma 6.3 remains true if we fix this value of $(\mathbf{a}, \mathbf{l}^L, \mathbf{l}^R) = (\mathbf{a}_0, \mathbf{l}_0^L, \mathbf{l}_0^R)$ instead of sampling $(\mathbf{a}, \mathbf{l}^L, \mathbf{l}^R)$ from \mathfrak{A} .*

Proof. Lemma 6.3 is a statement about the conditional law of (h^b, η^b) given $(\mathbf{a}, \mathbf{l}^L, \mathbf{l}^R)$ which holds $\mathfrak{m}(\mathfrak{A})^{-1} \mathfrak{m}|_{\mathfrak{A}}$ -a.s. Letting \mathfrak{A} vary shows that the statement of the lemma holds for \mathfrak{m} -a.e. triple $(\mathbf{a}_0, \mathbf{l}_0^L, \mathbf{l}_0^R) \in (0, \infty)^3$. Since the areas and left/right boundary lengths of the beads of $\mathcal{S}_{0,\infty}$ are a Poisson point process with intensity measure \mathfrak{m} and by Lemma 3.2, we infer that \mathfrak{m} is mutually absolutely continuous with respect to Lebesgue measure on $(0, \infty)^3$. The statement of the lemma follows. \square

Using Lemma 6.3, we see that the conditions in Theorem 6.2 actually imply a slightly stronger set of conditions after possibly re-choosing (h^b, η^b, Φ^b) .

Lemma 6.5. *In the statement of Theorem 6.2, we can choose the pair (h^b, η^b) and the homeomorphism Φ^b in condition 2 in such a way that Φ^b is conformal on each connected component of $\mathbb{H} \setminus \eta^b([0, t])$ and pushes forward $h^b|_{\mathbb{H} \setminus \eta^b([0, t])}$ to $\tilde{h}^b|_{\mathbb{H} \setminus \tilde{\eta}^b([0, t])}$ via the γ -LQG coordinate change formula.*

Proof. By condition 1 in Theorem 6.2 and Lemma 6.3, each applied with $t = \mathbf{a}$, the collection of singly-marked quantum surfaces obtained by restricting \tilde{h}^b to the bubbles cut out by $\tilde{\eta}^b$ and the collection of singly-marked quantum surfaces obtained by restricting h^b to the bubbles cut out by η^b have the same conditional law given Z^b . Hence we can find a coupling of $(\tilde{h}^b, \tilde{\eta}^b)$ and (h^b, η^b) with another pair $(\hat{h}^b, \hat{\eta}^b) \stackrel{d}{=} (h^b, \eta^b)$ such that Z^b is the left/right boundary length process of $(\hat{h}^b, \hat{\eta}^b)$ and the quantum surface obtained by restricting \hat{h}^b to each of the bubbles disconnected from ∞ by $\hat{\eta}^b$ agrees with the quantum surface obtained by restricting \tilde{h}^b to the corresponding bubble disconnected from ∞ by $\tilde{\eta}^b$. We will prove that the conditions of Theorem 6.2 plus the additional condition in the statement of the lemma are satisfied with $(\hat{h}^b, \hat{\eta}^b)$ in place of (h^b, η^b) .

The discussion just above Lemma 6.3 together with the peanosphere construction (c.f. [DMS14, Figure 1.18, Line 3]) imply that the curve-decorated topological space (\mathbb{H}, η^b) is a.s. determined by Z^b modulo a curve-preserving homeomorphism which also preserves the γ -quantum boundary length measure on each connected component of $\mathbb{H} \setminus \eta^b$. Hence condition 2 in Theorem 6.2 implies that a.s. $(\mathbb{H}, \tilde{\eta}^b)$, (\mathbb{H}, η^b) , and $(\mathbb{H}, \hat{\eta}^b)$ differ by curve-preserving homeomorphisms which also preserve the γ -quantum boundary length measure on each complementary connected component of the curve. Let $\tilde{\Phi}^b : \mathbb{H} \rightarrow \mathbb{H}$ be such a curve- and boundary measure-preserving homeomorphism from $(\mathbb{H}, \hat{\eta}^b)$ to $(\mathbb{H}, \tilde{\eta}^b)$.

By definition of $\hat{\eta}^b$, for each connected component \hat{D} of $\mathbb{H} \setminus \hat{\eta}^b$, there exists a conformal map $f_{\hat{D}} : \hat{D} \rightarrow \tilde{\Phi}^b(\hat{D})$ which pushes forward $\hat{h}^b|_{\hat{D}}$ to $\tilde{h}^b|_{\tilde{\Phi}^b(\hat{D})}$ via the γ -LQG coordinate change formula. The map $f_{\hat{D}}$ preserves the γ -quantum boundary length measure on $\partial \hat{D}$, so $\tilde{\Phi}^b|_{\partial \hat{D}} = f_{\hat{D}}|_{\partial \hat{D}}$. Therefore, if we re-define $\tilde{\Phi}^b$ to be equal to $f_{\hat{D}}$ on each such connected component \hat{D} , then $\tilde{\Phi}^b$ remains a curve- and boundary measure-preserving homeomorphism and pushes forward each $\hat{h}^b|_{\hat{D}}$ to $\tilde{h}^b|_{\tilde{\Phi}^b(\hat{D})}$ via the γ -LQG coordinate change formula.

Since condition 1 depends on the pair (h^b, η^b) only via the function Z^b , it remains only to check the boundary length consistency statement in condition 2 with $(\tilde{h}^b, \tilde{\eta}^b, \tilde{\Phi}^b)$ in place of (h^b, η^b, Φ^b) . To see this, we observe that condition 2 in Theorem 6.2 (together with the analogous property for (h^b, η^b)) imply that a.s. $\nu_{\tilde{h}^b}(\tilde{\eta}^b([0, t]) \cap \tilde{\eta}^b([t, a])) = 0$ for each $t \in [0, a]$. Hence $\nu_{\tilde{h}^b}$ -a.e. point on the boundary of the unbounded connected component of $\tilde{\eta}^b([0, t])$ lies on the boundary of a connected component of $\mathbb{H} \setminus \tilde{\eta}^b$. Since $\tilde{\Phi}^b$ preserves the γ -quantum boundary length measure on each such component, we conclude. \square

Henceforth assume that (h^b, η^b) and Φ^b have been chosen as in Lemma 6.5. Recall from the discussion above Lemma 6.3 that we have coupled $(\tilde{h}^b, \tilde{\eta}^b)$ and (h^b, η^b) with (h, η') in such a way that a.s. $(\mathcal{B}, \eta_{0,\infty,\mathcal{B}}(\cdot - \underline{T}))$ and $((\mathbb{H}, h^b, 0, \infty), \eta^b)$ agree as curve-decorated quantum surfaces and (h, η') is conditionally independent from $(\tilde{h}^b, \tilde{\eta}^b)$ and (h^b, η^b) given $(\mathcal{B}, \eta_{0,\infty,\mathcal{B}}(\cdot - \underline{T}))$. We now use this coupling to construct a pair $(\tilde{h}, \tilde{\eta}')$ which we will eventually show satisfies the conditions of Theorem 2.4.

Let $(\eta')^b : [0, a] \rightarrow \mathbb{H}$ be the space-filling curve which is the image of $\eta'_\mathcal{B}(\cdot - \underline{T})$ under the embedding h^b of \mathcal{B} into $(\mathbb{H}, 0, \infty)$. Let $(\tilde{\eta}')^b := \Phi^b \circ (\eta')^b : [0, a] \rightarrow \mathbb{H}$. By the condition on Φ^b in Lemma 6.5 and our choice of coupling, the conditional law of $(\tilde{\eta}')^b$ given (h^b, η^b) is obtained by replacing each of the constant increments $\tilde{\eta}^b|_{[\sigma^b(t), \tau^b(t)]}$ by an independent space-filling SLE $_{\kappa'}$ loop based at $\tilde{\eta}^b(t)$ in the bubble disconnected from ∞ by $\tilde{\eta}^b$ at time $\sigma^b(t)$ based at $\tilde{\eta}^b(t)$, parameterized by γ -quantum mass with respect to \tilde{h}^b .

Let $\tilde{\mathcal{C}} = (\mathbb{C}, \tilde{h}, 0, \infty)$ be the quantum surface obtained from the γ -quantum cone \mathcal{C} by replacing the bead \mathcal{B} with the quantum surface $\tilde{\mathcal{B}} := (\mathbb{H}, \tilde{h}^b, 0, \infty) \stackrel{d}{=} \mathcal{B}$, equivalently the quantum surface obtained by conformally welding together $(\mathbb{H}, \tilde{h}^b, 0, \infty)$ and the complementary sub-surface $\mathcal{C} \setminus \mathcal{B}$ according to γ -quantum lengths along their boundaries. The boundary of the region $\eta'([T, \bar{T}])$ in \mathbb{C} which parameterizes \mathcal{B} is a union of two non-crossing SLE $_{\kappa}$ segments, for $\kappa = 16/\kappa' \in (0, 4)$, so is a.s. conformally removable, so there is a unique way to perform this conformal welding and $(\mathbb{C}, \tilde{h}, 0, \infty)$ is well-defined. By our choice of coupling, (h, η') is conditionally independent from $(\tilde{h}^b, \tilde{\eta}^b)$ given \mathcal{B} , so $(\mathbb{C}, \tilde{h}, 0, \infty)$ has the law of a γ -quantum cone.

Let $\tilde{\eta}' : \mathbb{R} \rightarrow \mathbb{C}$ be the curve which is the image under the conformal welding map of the concatenation of $\eta'_{\mathcal{C} \setminus \mathcal{B}}|_{(-\infty, \underline{T})}$, $\tilde{\eta}'_{\tilde{\mathcal{B}}}(\cdot + \underline{T})$, and $\eta'_{\mathcal{C} \setminus \mathcal{B}}|_{[\bar{T}, \infty)}$. Then $\tilde{\eta}'$ is a space-filling curve from ∞ to ∞ parameterized by γ -quantum mass with respect to \tilde{h} . The main input in the proof of Theorem 6.2 is the following lemma.

Lemma 6.6. *The conditions of Theorem 2.4 are satisfied for the pairs (h, η') and $(\tilde{h}, \tilde{\eta}')$ defined above. Hence $(\tilde{h}, \tilde{\eta}')$ is an embedding into $(\mathbb{C}, 0, \infty)$ of a γ -quantum cone together with an independent whole-plane space-filling SLE $_{\kappa'}$ from ∞ to ∞ parameterized by γ -quantum mass with respect to \tilde{h} .*

For the proof of Lemma 6.6, we recall the definitions of the quantum surfaces $\mathcal{S}_{a,b}$ for $a, b \in \mathbb{R} \cup \{-\infty, \infty\}$ and the times $\sigma(t) = \sigma_{0,\infty}(t)$ and $\tau(t) = \tau_{0,\infty}(t)$ for $t \geq 0$ from above, and define the quantum surfaces $\tilde{\mathcal{S}}_{a,b}$ for $a < b$ obtained by restricting \tilde{h} to $\tilde{\eta}'([a, b])$, as in (3.2).

We start by defining the homeomorphism $\Phi : \mathbb{C} \rightarrow \mathbb{C}$ appearing in condition 2 in Theorem 2.4. Let $\Phi_{\mathcal{C}} : \mathcal{C} \rightarrow \tilde{\mathcal{C}}$ be the homeomorphism which is given by the identity map on the sub-surface of \mathcal{C} parameterized by $\mathcal{C} \setminus \eta'_{\mathcal{C}}([T, \bar{T}])$ and which is given by the homeomorphism Φ^b from Lemma 6.5 (viewed as a map between quantum surfaces) on the sub-surface of \mathcal{C} parameterized by $\eta'_{\mathcal{C}}([T, \bar{T}])$. Let $\Phi : \mathbb{C} \rightarrow \mathbb{C}$ be the map corresponding to $\Phi_{\mathcal{C}}$ for the embeddings h and \tilde{h} . The above construction of $\tilde{\eta}'$ implies that $\Phi \circ \eta' = \tilde{\eta}'$. By condition 2 in Theorem 6.2 and our construction of \tilde{h} , it is a.s. the case that Φ pushes forward the γ -quantum length measure with respect to h^b on the boundary of the unbounded connected component of $\mathbb{H} \setminus \tilde{\eta}^b([0, t])$ to the γ -quantum length measure with respect to \tilde{h}^b on the boundary of the unbounded connected component of $\mathbb{H} \setminus \tilde{\eta}^b([0, t])$ for each $t \in [0, a] \cap \mathbb{Q}$. Thus condition 2 in Theorem 2.4 is satisfied.

We note that, due to our choice of the triple (h^b, η^b, Φ^b) from Lemma 6.5 and the conformal removability of $\partial\eta'([T, \bar{T}])$, the map Φ is in fact conformal on $\mathbb{C} \setminus \eta_{0,\infty}([T, \bar{T}])$, with $\eta_{0,\infty}$ the future chordal SLE $_{\kappa'}$ curve obtained by skipping the bubbles filled in by η' during the time interval $[0, \infty)$ as above and pushes forward $h|_{\mathbb{C} \setminus \eta_{0,\infty}([T, \bar{T}])}$ to the corresponding restriction of \tilde{h} via the γ -LQG coordinate change formula (2.2). In particular, we have the equalities of curve-decorated quantum surfaces

$$\begin{aligned} (\mathcal{S}_{-\infty, \underline{T}}, \eta'_{\mathcal{S}_{-\infty, \underline{T}}}) &= (\tilde{\mathcal{S}}_{-\infty, \underline{T}}, \tilde{\eta}'_{\tilde{\mathcal{S}}_{-\infty, \underline{T}}}), & (\mathcal{S}_{\bar{T}, \infty}, \eta'_{\mathcal{S}_{\bar{T}, \infty}}) &= (\tilde{\mathcal{S}}_{\bar{T}, \infty}, \tilde{\eta}'_{\tilde{\mathcal{S}}_{\bar{T}, \infty}}), \quad \text{and} \\ (\mathcal{S}_{\sigma(t), \tau(t)}, \eta'_{\mathcal{S}_{\sigma(t), \tau(t)}}) &= (\tilde{\mathcal{S}}_{\sigma(t), \tau(t)}, \tilde{\eta}'_{\tilde{\mathcal{S}}_{\sigma(t), \tau(t)}}), \quad \forall t \geq 0. \end{aligned} \tag{6.6}$$

To check condition 1 in Theorem 2.4, we need the following lemma.

Lemma 6.7. *Define the σ -algebras $\tilde{\mathcal{F}}_{-\infty,t}$ and $\tilde{\mathcal{F}}_{t,\infty}$ as in Theorem 2.4 for $(\tilde{h}, \tilde{\eta}')$ as above. For each $t \geq 0$, we have $\tilde{\mathcal{F}}_{t,\infty} = \sigma((Z - Z_t)|_{[t,\infty)})$. Furthermore, \bar{T} is a stopping time for Z and for each $t \geq 0$, $\tilde{\mathcal{F}}_{-\infty,t \wedge \bar{T}} = \sigma((Z - Z_{t \wedge \bar{T}})|_{(-\infty, t \wedge \bar{T}]})$.*

Proof. For $t \geq 0$, each $\pi/2$ -cone interval $[v_Z(s), s]$ for Z which is contained in $[t, \infty)$ is contained in the maximal $\pi/2$ -cone interval $[\sigma(r), \tau(r)]$ in $[0, \infty)$ for any $r \in (v_Z(s), s)$. By (6.6), a.s. $\mathcal{S}_{v_Z(s),s} = \tilde{\mathcal{S}}_{v_Z(s),s}$ for each such $\pi/2$ -cone interval $[v_Z(s), s]$. Since the peanosphere Brownian motion determines the quantum surfaces corresponding to segments of η' in a local manner, we infer that the quantum surfaces $\tilde{\mathcal{S}}_{v_Z(s),s}$ as s ranges over all $\pi/2$ -cone intervals contained in $[t, \infty)$ are a.s. determined by $(Z - Z_t)|_{[t,\infty)}$, which gives the first statement of the lemma.

The time \bar{T} is the smallest $t \geq 0$ such that the following is true: L and R attain a simultaneous running infimum relative to time 0 at time t and if t' denotes the last time strictly before t at which L and R attain a simultaneous running infimum, then the vector $(t - t', L_{t'} - L_t, R_{t'} - R_t)$ belongs to \mathfrak{A} . Hence \bar{T} is a stopping time for Z (and hence also for $\{\tilde{\mathcal{F}}_{-\infty,t}\}_{t \geq 0}$).

To prove the formula for $\tilde{\mathcal{F}}_{-\infty,t \wedge \bar{T}}$, let $[v_Z(s), s]$ be a $\pi/2$ -cone interval for Z in $(-\infty, t \wedge \bar{T})$. Since L and R attain a simultaneous running infimum relative to time 0 at time \underline{T} , it follows that \underline{T} is a $\pi/2$ -cone time for Z with $v_Z(\underline{T}) < 0$. Since \bar{T} is the next time after \underline{T} at which L and R attain a simultaneous running infimum relative to time 0, it cannot be the case that $s \in (\underline{T}, t \wedge \bar{T})$ and $v_Z(s) < 0$. Since two $\pi/2$ -cone intervals for Z are either nested or disjoint, either $[v_Z(s), s] \subset (-\infty, \underline{T})$ or $[v_Z(s), s] \subset [\underline{T}, t \wedge \bar{T}]$. In the latter case, $[v_Z(s), s]$ is contained in a maximal $\pi/2$ -cone time for Z in $[0, \infty)$. By (6.6), a.s. $\mathcal{S}_{v_Z(s),s} = \tilde{\mathcal{S}}_{v_Z(s),s}$ for each such $\pi/2$ -cone interval $[v_Z(s), s]$ and we conclude as in the case of $\tilde{\mathcal{F}}_{t,\infty}$. \square

Lemma 6.8. *For each $t \in \mathbb{R}$, the σ -algebras $\tilde{\mathcal{F}}_{-\infty,t}$ and $\tilde{\mathcal{F}}_{t,\infty}$ from the statement of Theorem 2.4 are independent.*

Proof. If $t < 0$, then the curve-decorated quantum surface $(\tilde{\mathcal{S}}_{t,\infty}, \tilde{\eta}'_{\tilde{\mathcal{S}}_{t,\infty}})$, which determines $\tilde{\mathcal{F}}_{t,\infty}$, is independent from $\tilde{\mathcal{F}}_{-\infty,t}$. Hence we can restrict attention to the case when $t \geq 0$.

By Lemma 6.7, for $t \geq 0$ the conditional law of $(Z - Z_t)|_{[t,\infty)}$ given $\tilde{\mathcal{F}}_{-\infty,t}$ on the event $\{t \leq \bar{T}\}$ is the same as its marginal law. Since $(\tilde{\mathcal{S}}_{\underline{T},\bar{T}}, \tilde{\eta}'_{\tilde{\mathcal{S}}_{\underline{T},\bar{T}}})$ is conditionally independent from (h, η') given $(\mathcal{S}_{\underline{T},\bar{T}}, \eta'_{\mathcal{S}_{\underline{T},\bar{T}}})$, the conditional law of $(Z - Z_t)|_{[t,\infty)}$ given $(Z - Z_t)|_{(-\infty,t]}$ and $(\tilde{\mathcal{S}}_{\underline{T},\bar{T}}, \tilde{\eta}'_{\tilde{\mathcal{S}}_{\underline{T},\bar{T}}})$ on the event $\{t > \bar{T}\}$ is the same as its marginal law. The σ -algebra $\tilde{\mathcal{F}}_{-\infty,t}$ is clearly contained in the σ -algebra generated by $(Z - Z_t)|_{(-\infty,t]}$ and $(\tilde{\mathcal{S}}_{\underline{T},\bar{T}}, \tilde{\eta}'_{\tilde{\mathcal{S}}_{\underline{T},\bar{T}}})$. Hence the conditional law of $(Z - Z_t)|_{[t,\infty)}$ given $\tilde{\mathcal{F}}_{-\infty,t}$ is the same as its marginal law. By Lemma 6.7, $\tilde{\mathcal{F}}_{t,\infty} = \sigma((Z - Z_t)|_{[t,\infty)})$ so the statement of the lemma follows. \square

We now check the last remaining condition of Theorem 2.4.

Lemma 6.9. *For each $t \in \mathbb{R}$, the future quantum surface $\tilde{\mathcal{S}}_{t,\infty}$ has the law of a $\frac{3\gamma}{2}$ -quantum wedge and is independent from $\tilde{\mathcal{F}}_{-\infty,t}$.*

Proof. It is clear that the statement of the lemma holds for $t < 0$. To check the condition for $t > 0$, we define for $r \in \mathbb{R}$ the σ -algebra

$$\tilde{\mathcal{G}}_r := \tilde{\mathcal{F}}_{-\infty,r} \vee \sigma(P_{r,\infty}) \quad (6.7)$$

as in Lemma 3.13. Then both \underline{T} and \bar{T} are $\tilde{\mathcal{G}}_0$ -measurable. Also define for $t \geq 0$

$$\underline{\tau}(t) := (\tau(t) \vee \underline{T}) \wedge t, \quad (6.8)$$

where $\tau(t) = \tau_{0,\infty}(t)$ is as above. The time $\underline{\tau}(t)$ is equal to \underline{T} , the left endpoint of the interval on which $P_{t,\infty}$ is constant whose right endpoint is \bar{T} , or t according to whether $t < \underline{T}$, $t \in [\underline{T}, \bar{T}]$, or $t > \bar{T}$. Hence $\underline{\tau}(t) \in \tilde{\mathcal{G}}_t$.

The beaded quantum surface $\tilde{\mathcal{S}}_{t,\infty}$ is the concatenation of the beaded quantum surfaces $\tilde{\mathcal{S}}_{t,\underline{\tau}(t)}$, $\tilde{\mathcal{S}}_{\underline{\tau}(t),\overline{\tau}\vee t}$, and $\tilde{\mathcal{S}}_{\overline{\tau}\vee t,\infty}$, with $\underline{\tau}(t)$ as in (6.8) (note that the first two of these quantum surfaces are degenerate if $t > \overline{\tau}$). We will now consider the conditional laws of these quantum surfaces given the σ -algebra $\tilde{\mathcal{G}}_t$ of (6.7).

Since $\tilde{\eta}'([t, \underline{\tau}(t)])$ is either a single point or is contained in either $\tilde{\eta}'([t, \underline{\tau}(t)])$ or in one of the bubbles cut out by $\tilde{\eta}_{0,\infty}$, (6.6) implies that a.s. $\tilde{\mathcal{S}}_{t,\underline{\tau}(t)} = \mathcal{S}_{t,\underline{\tau}(t)}$. The conditional law of this quantum surface given $\tilde{\mathcal{G}}_t$ is that of an ordered collection of beads of a $\frac{3\gamma}{2}$ -quantum wedge with areas and left/right boundary lengths specified by $P_{t,\infty}|_{[t,\underline{\tau}(t)]}$.

The quantum surface $\tilde{\mathcal{S}}_{\underline{\tau}(t),\overline{\tau}\vee t}$ is equal to $\tilde{\mathcal{S}}_{\underline{\tau},\overline{\tau}}$ if $t < \underline{\tau}$, to the quantum surface $\tilde{\mathcal{W}}_t^b$ in Theorem 6.2 if $t \in [\underline{\tau}, \overline{\tau}]$, or to a single point if $t > \overline{\tau}$. By condition 1 in Theorem 6.2 and Lemma 6.7, we infer that the conditional law of $\tilde{\mathcal{S}}_{\underline{\tau}(t),\overline{\tau}\vee t}$ given $\tilde{\mathcal{G}}_t$ and $\tilde{\mathcal{S}}_{t,\underline{\tau}(t)}$ is that of a single bead of a $\frac{3\gamma}{2}$ -quantum wedge with area and left/right boundary lengths given by the value of $P_{t,\infty}$ on $[\underline{\tau}(t), \overline{\tau}]$ (or a single point, if this interval is empty). By combining this with the previous paragraph, we find that the conditional law of $\tilde{\mathcal{S}}_{t,\overline{\tau}\vee t}$ given $\tilde{\mathcal{G}}_t$ is that of an ordered collection of beads of a $\frac{3\gamma}{2}$ -quantum wedge with areas and left/right boundary lengths specified by $P_{t,\infty}|_{[t,\overline{\tau}\vee t]}$.

We have $\tilde{\mathcal{S}}_{\overline{\tau}\vee t,\infty} = \mathcal{S}_{\overline{\tau}\vee t,\infty}$, and the conditional law of this quantum surface given $\tilde{\mathcal{G}}_t$ and $\tilde{\mathcal{S}}_{t,\overline{\tau}\vee t}$ is that of a collection of independent beads of a $\frac{3\gamma}{2}$ -quantum wedge with areas and left/right boundary lengths specified by $P_{t,\infty}|_{[\overline{\tau}\vee t,\infty)}$. Hence the conditional law of $\tilde{\mathcal{S}}_{t,\infty}$ given $\tilde{\mathcal{G}}_t$ is that of an ordered collection of beads of a $\frac{3\gamma}{2}$ -quantum wedge with areas and left/right boundary lengths specified by $P_{t,\infty}$. This is the same as the conditional law of $\mathcal{S}_{t,\infty}$ given $P_{t,\infty}$. Averaging over all possible realizations of $P_{t,\infty}$ shows that the conditional laws of $\tilde{\mathcal{S}}_{t,\infty}$ and $\mathcal{S}_{t,\infty}$ given $\tilde{\mathcal{F}}_{-\infty,t}$ agree. That is, $\tilde{\mathcal{S}}_{t,\infty}$ is a $\frac{3\gamma}{2}$ -quantum wedge independent from $\tilde{\mathcal{F}}_{-\infty,t}$, as required. \square

Combining the above results shows that the conditions of Theorem 2.4 are satisfied for the pairs (h, η') and $(\tilde{h}, \tilde{\eta}')$ defined above, and the last statement of Lemma 6.6 follows from Theorem 2.4. \square

We can now conclude the proof of Theorem 6.2. The main remaining obstacle is to transfer from the case where $(\mathbf{a}, \mathbf{l}^L, \mathbf{l}^R)$ is random (which we have been considering throughout this subsection) to the case when $(\mathbf{a}, \mathbf{l}^L, \mathbf{l}^R)$ is deterministic.

Proof of Theorem 6.2. Let \mathbf{m} be the infinite measure on beads of a $\frac{3\gamma}{2}$ -quantum wedge and let $\mathfrak{A} \subset (0, \infty)^3$ be a Borel set such that $\mathbf{m}(\mathfrak{A})$ is finite and positive, as in the beginning of this subsection. Lemma 6.6 immediately implies the variant of Theorem 6.2 where $(\mathbf{a}, \mathbf{l}^L, \mathbf{l}^R)$ is sampled from $\mathbf{m}(\mathfrak{A})^{-1}\mathbf{m}|_{\mathfrak{A}}$ (i.e., the setting of most of this section).

Since $\mathbf{m}(\mathfrak{A})^{-1}\mathbf{m}|_{\mathfrak{A}}$ is mutually absolutely continuous with respect to Lebesgue measure on $(0, \infty)^3$ (this is immediate, e.g., from the peanosphere construction), we find that the theorem statement remains true if instead $(\mathbf{a}, \mathbf{l}^L, \mathbf{l}^R)$ is sampled from a probability measure P on $(0, \infty)^3$ which is mutually absolutely continuous with respect to Lebesgue measure on any given Borel subset of $(0, \infty)^3$.

Suppose now that $(\mathbf{a}, \mathbf{l}^L, \mathbf{l}^R)$ is deterministic, as in the theorem statement. For $t \in [0, \mathbf{a}]$, let $\tilde{\mathcal{W}}_t^b$ be the quantum surface parameterized by the unbounded connected component of $\mathbb{H} \setminus \tilde{\eta}^b([0, t])$, as in condition 1 in the theorem statement, and let \mathcal{W}_t^b be defined analogously with the pair (h^b, η^b) in place of the pair $(\tilde{h}^b, \tilde{\eta}^b)$. If $\epsilon > 0$ and we condition on $Z^b|_{[0, \epsilon]}$ and $\tau^b(\epsilon)$, then the conditions in the theorem statement are satisfied with embeddings into $(\mathbb{H}, 0, \infty)$ of the pair of curve-decorated quantum surfaces $(\tilde{\mathcal{W}}_\epsilon^b, \tilde{\eta}_{\tilde{\mathcal{W}}_\epsilon^b}^b)$ and $(\mathcal{W}_\epsilon^b, \eta_{\mathcal{W}_\epsilon^b}^b)$ in place of (h^b, η^b) and $(\tilde{h}^b, \tilde{\eta}^b)$; the restriction of Φ^b to the unbounded connected component of $\mathbb{H} \setminus \eta^b([0, t])$, post-composed and pre-composed with appropriate conformal maps, in place of Φ^b ; and the triple $(\mathbf{a} - \tau^b(\epsilon), L_\epsilon^b, R_\epsilon^b)$ in place of $(\mathbf{a}, \mathbf{l}^L, \mathbf{l}^R)$. By local absolute continuity of h with respect to an embedding into $(\mathbb{H}, 0, \infty)$ of a bead of a $\frac{3\gamma}{2}$ -quantum wedge with random area and boundary length, we infer that the joint law of $(\mathbf{a} - \tau^b(\epsilon), L_\epsilon^b, R_\epsilon^b)$ is mutually absolutely continuous with respect to Lebesgue measure on $(0, \mathbf{a} - \epsilon) \times (0, \infty)^2$. Applying the theorem in the case when $(\mathbf{a}, \mathbf{l}^L, \mathbf{l}^R)$ is random and sending $\epsilon \rightarrow 0$ now yields the theorem in the case when $(\mathbf{a}, \mathbf{l}^L, \mathbf{l}^R)$ is deterministic. \square

7 Characterizations of SLE_6 on a Brownian surface

7.1 Space-filling and chordal statements

In the special case when $\kappa' = 6$ (so $\gamma = \sqrt{8/3}$), the $\sqrt{8/3}$ -LQG metric on a $\sqrt{8/3}$ -LQG surface is well-defined (recall Section 2.2.4). In this case, we can re-phrase Theorems 2.4 and 6.2 in terms of metric measure spaces rather than quantum surfaces.

For the theorem statements in this subsection, we recall the definition of the internal metric: if (X, d) is a metric space and $Y \subset X$, the internal metric d_Y of d on Y is defined by setting $d_Y(y_1, y_2)$ for $y_1, y_2 \in Y$ to be the infimum of the d -lengths of all paths in Y from y_1 to y_2 . Note that $d_Y(y_1, y_2)$ may be infinite.

In order to make sense of random metric measure spaces, we endow the space of compact finite metric measure spaces with the *Gromov-Prokhorov* or *Gromov-weak* topology [GPW09], which is the same topology used in [MS15a]. This is the topology whereby two metric measure spaces are close if they can be isometrically embedded into a common metric space such that the corresponding measures are close in the Prokhorov distance. In the case of locally compact, locally finite metric length spaces (such as the Brownian plane), we instead use the local variant of the Gromov-Prokhorov topology, the *Gromov-vague* topology, which is defined in [ALW16].

We define a *singly (resp. doubly) marked Brownian disk* to be a Brownian disk together with one (resp. two) marked points sampled uniformly from its natural boundary measure. A doubly marked Brownian disk has a notion of left and right quantum boundary lengths, corresponding to the lengths of the clockwise and counterclockwise boundary arcs between the two marked points (we take the marked points to be ordered). In the special case when $\gamma = \sqrt{8/3}$, a single bead of a $(\frac{3\gamma}{2} = \sqrt{6})$ -quantum wedge is the same as a quantum disk with two marked boundary points (c.f. Section 2.2.2), which in turn is equivalent as a metric measure space to a doubly marked Brownian disk [MS16b, Corollary 1.5]. We also recall that by [MS16b, Corollary 1.5], the $\sqrt{8/3}$ -quantum cone is equivalent to the Brownian plane [CL14].

We first state a metric space version of Theorem 2.4, which characterizes a coupling of an instance $(\tilde{X}, \tilde{d}, \tilde{\mu}, \tilde{x})$ of the Brownian plane (equipped with its natural metric, area measure, and marked point), a curve $\tilde{\eta}' : \mathbb{R} \rightarrow \tilde{X}$ with $\tilde{\eta}'(0) = 0$, and a correlated two-dimensional Brownian motion Z with variances and covariance as in (1.2) for $\kappa' = 6$. For the statement of the theorem, we need to define σ -algebras analogous to the σ -algebras $\tilde{\mathcal{F}}_{a,b}$ in Theorem 2.4 (but in terms of the metric measure space rather than quantum surface structure). For $a, b \in \mathbb{R} \cup \{-\infty, \infty\}$, with $a < b$, let $\tilde{X}_{a,b}$ be the interior of $\tilde{\eta}'([a, b])$ and let $\tilde{d}_{a,b}$ be the internal metric of \tilde{d} on $\tilde{X}_{a,b}$. Also let $\tilde{\mathcal{F}}_{a,b}^m$ be the σ -algebra generated by $(Z - Z_a)|_{[a,b]}$ (or $(Z - Z_b)|_{[a,b]}$ if $a = -\infty$) and the pointed metric measure spaces $(\tilde{X}_{v_Z(s), s}, \tilde{d}_{v_Z(s), s}, \tilde{\mu}|_{\tilde{X}_{v_Z(s), s}}, \tilde{\eta}'(s))$ where s ranges over all $\pi/2$ -cone times for Z which are maximal in some interval contained in (a, b) with rational endpoints. Here we recall that $v_Z(s)$ is the start time of the $\pi/2$ -cone excursion (Definition 2.2).

Theorem 7.1 (Whole-plane space-filling SLE characterization, metric version). *Let $\kappa' = 6$ and $\gamma = \sqrt{8/3}$. Suppose that $(\tilde{X}, \tilde{d}, \tilde{\mu}, \tilde{x})$ is a coupling of an instance of the Brownian plane (equipped with its natural metric, area measure, and marked point), a curve $\tilde{\eta}' : \mathbb{R} \rightarrow \tilde{X}$ with $\tilde{\eta}'(0) = 0$, and a correlated two-dimensional Brownian motion Z with variances and covariance as in (1.2) for $\kappa' = 6$. Assume that the following conditions are satisfied.*

1. (Markov property) *For each $t \in \mathbb{R}$, the σ -algebras $\tilde{\mathcal{F}}_{-\infty, t}^m$ and $\tilde{\mathcal{F}}_{t, \infty}^m$ defined just above are independent. Furthermore, for each $t \in \mathbb{R}$ the ordered collection of connected components of the metric measure space $(\tilde{X}_{t, \infty}, \tilde{d}_{t, \infty}, \tilde{\mu}|_{\tilde{X}_{t, \infty}})$ (in the order they are filled in by $\tilde{\eta}'$) has the same law as the beads of a $\sqrt{6}$ -quantum wedge equipped with their $\sqrt{8/3}$ -LQG metrics and area measures, i.e. its law is that of a Poissonian collection of doubly marked Brownian disks, lying at infinite internal distance from each other, with areas and left/right boundary lengths specified by the times when $(Z - Z_t)|_{[t, \infty)}$ attains a simultaneous running infimum relative to time t and the values of the coordinates of Z at these times. Furthermore, the ordered collection of connected components of the metric measure space $(\tilde{X}_{t, \infty}, \tilde{d}_{t, \infty}, \tilde{\mu}|_{\tilde{X}_{t, \infty}})$ is independent from $\tilde{\mathcal{F}}_{-\infty, t}^m$.*
2. (Topology and consistency) *The curve-decorated topological space $(\tilde{X}, \tilde{\eta}')$ is equivalent to the infinite-volume peanosphere generated by Z . Equivalently, if $((\mathbb{C}, h, 0, \infty), \eta')$ is the pair consisting of a γ -quantum*

cone and an independent space-filling $\text{SLE}_{\kappa'}$ from ∞ to ∞ parameterized by γ -quantum mass with respect to h which is determined by Z via [DMS14, Theorem 1.14], then there is a homeomorphism $\Phi : \mathbb{C} \rightarrow \tilde{X}$ with $\Phi \circ \eta' = \tilde{\eta}'$. Moreover, Φ a.s. pushes forward the $\sqrt{8/3}$ -quantum length measure on $\partial\eta'([t, \infty))$ with respect to h to the natural length measure on $\partial\tilde{\eta}'([t, \infty))$ (which is defined via the length measures on the Brownian disks which are the connected components of $\tilde{X}_{t, \infty}$).

Then $(\tilde{X}, \tilde{d}, \tilde{\mu}, \tilde{\eta}')$ is equivalent (as a curve-decorated metric measure space) to a $\sqrt{8/3}$ -quantum cone decorated by an independent whole-plane space-filling $\text{SLE}_{\kappa'}$ from ∞ to ∞ parameterized by $\sqrt{8/3}$ -quantum mass with respect to h . In fact, the map Φ of condition 2 is a.s. an isometry.

We next state a metric space version of Theorem 6.2.

Theorem 7.2 (Chordal SLE characterization on a quantum disk, metric version). *Suppose $\kappa' = 6$ and $\gamma = \sqrt{8/3}$. Let $(\mathbf{a}, \mathfrak{l}^L, \mathfrak{l}^R) \in (0, \infty)^3$ and suppose we are given a coupling of a doubly-marked Brownian disk $(\tilde{X}^b, \tilde{d}^b, \tilde{\mu}^b, \tilde{x}, \tilde{y})$ with area \mathbf{a} and left/right boundary lengths \mathfrak{l}^L and \mathfrak{l}^R , a random continuous curve $\tilde{\eta}^b : [0, \mathbf{a}] \rightarrow \tilde{X}^b$ from \tilde{x} to \tilde{y} , parameterized by the $\tilde{\mu}^b$ -mass it disconnects from ∞ (Definition 6.1), and a random discontinuous process $Z^b = (L^b, R^b)$ as described in Section 6 for $\kappa' = 6$. Assume that the following conditions are satisfied.*

1. (Laws of complementary connected components) For $t \in [0, \mathbf{a}]$ let $\tilde{\mathcal{U}}_t^b$ be the collection of singly marked metric measure spaces of the form $(U, \tilde{d}_U^b, \tilde{\mu}_U^b|_U, \tilde{x}_U)$ where U is a connected component of $\tilde{X}^b \setminus \tilde{\eta}^b([0, t])$, \tilde{d}_U^b is the internal metric of \tilde{d}^b on U , and \tilde{x}_U is the point where $\tilde{\eta}^b$ finishes tracing ∂U . If we condition on $Z^b|_{[0, t]}$ and the time $\tau^b(t)$ from (6.2), then the conditional law of $\tilde{\mathcal{U}}_t^b$ is that of a collection of independent singly marked Brownian disks with areas and boundary lengths specified as follows. The elements of $\tilde{\mathcal{U}}_t^b$ corresponding to the connected components of $\tilde{X}^b \setminus \tilde{\eta}^b([0, t])$ which do not have the target point $\tilde{\eta}^b(\mathbf{a})$ on their boundaries are in one-to-one correspondence with the intervals of time in $[0, \tau^b(t)]$ on which Z^b is constant, and their areas and boundary lengths are determined by Z^b as in (6.2). The element of $\tilde{\mathcal{U}}_t^b$ corresponding to the connected component of $\tilde{X}^b \setminus \tilde{\eta}^b([0, t])$ with $\tilde{\eta}^b(\mathbf{a})$ on its boundary has area $\mathbf{a} - \tau^b(t)$ and boundary length $L_t^b + R_t^b$.
2. (Topology and consistency) The topology of $(\tilde{X}^b, \tilde{\eta}^b)$ is determined by Z^b in the same manner as the topology of a chordal SLE_6 on a doubly marked quantum disk, i.e. there is a pair $((\mathbb{H}, h^b, 0, \infty), \eta^b)$ consisting of a doubly marked quantum disk with area \mathbf{a} and left/right boundary lengths \mathfrak{l}^L and \mathfrak{l}^R and an independent chordal SLE_6 from 0 to ∞ in \mathbb{H} parameterized by the μ_{h^b} -mass it disconnects from ∞ and a homeomorphism $\Phi^b : \mathbb{H} \rightarrow \tilde{X}^b$ with $\Phi^b \circ \eta^b = \tilde{\eta}^b$. Moreover, for each $t \in [0, \mathbf{a}] \cap \mathbb{Q}$, Φ^b a.s. pushes forward the $\sqrt{8/3}$ -quantum length measure with respect to h^b on the boundary of the unbounded connected component of $\mathbb{H} \setminus \eta^b([0, t])$ to the natural boundary length measure on the connected component of $\tilde{X}^b \setminus \tilde{\eta}^b([0, t])$ with $\tilde{\eta}^b(\mathbf{a})$ on its boundary (which is well-defined since we know the internal metric on this component is that of a Brownian disk).

Then $(\tilde{X}^b, \tilde{d}^b, \tilde{\mu}^b, \tilde{\eta}^b)$ is equivalent (as a curve-decorated metric measure space) to a doubly-marked quantum disk with area \mathbf{a} and left/right boundary lengths \mathfrak{l}_L and \mathfrak{l}_R equipped with its $\sqrt{8/3}$ -LQG area measure and metric together with an independent chordal SLE_6 from 0 to ∞ in \mathbb{H} parameterized by the $\sqrt{8/3}$ -quantum mass it disconnects from ∞ .

In the remainder of this section we will prove Theorems 7.1 and 7.2. In Section 7.4, we will also deduce a variant of Theorem 7.2 where we parameterize by quantum natural time instead of disconnected quantum area. In fact, this is the precise characterization statement which we will make use of in [GM17b].

We know from [MS16c, Theorem 1.4] that the metric measure space structure and the quantum surface structure of a $\sqrt{8/3}$ -LQG surface a.s. determine each other, i.e. each is a.s. given by a deterministic measurable function of the other. However, this together with Theorem 2.4 does not immediately imply Theorem 7.1 for the following reason. The deterministic function in [MS16c, Theorem 1.4] is allowed to depend on the particular law of the quantum surface, so it is a priori possible, e.g., that two quantum surfaces with the same metric measure space structure are not equivalent if they have different laws. The hypotheses of Theorem 7.1 only tell us about the laws of the metric measure space structures of certain quantum surfaces, so

we cannot immediately conclude anything about the conformal structures of these quantum surfaces. Similar considerations apply in the setting of Theorem 7.2. We will get around this difficulty by comparing the quantum surfaces parameterized by certain filled metric balls and using local absolute continuity.

Throughout this section, we will use the following notation. For a GFF-type distribution h on a domain $D \subset \mathbb{C}$ we write \mathfrak{d}_h , μ_h , and ν_h , respectively, for the $\sqrt{8/3}$ -LQG metric, area measure, and length measure induced by h . For $r > 0$ and $z \in D$, we write $B_r(z; \mathfrak{d}_h)$ for the closed \mathfrak{d}_h -ball of radius r centered at z . If $D \neq \mathbb{C}$ and $B_r(z; \mathfrak{d}_h)$ does not intersect ∂D (resp. if $D = \mathbb{C}$), we define the filled metric ball $B_r^\bullet(z; \mathfrak{d}_h)$ to be the union of $B_r(z; \mathfrak{d}_h)$ and the set of points which it disconnects from ∂D (resp. ∞).

We start in Section 7.2 by proving some general facts about the quantum surfaces $(B_r^\bullet(z; \mathfrak{d}_h), h|_{B_r^\bullet(z; \mathfrak{d}_h)})$ parameterized by filled metric and their corresponding metric measure space structures for general GFF-type distributions h . In particular, we will show that the quantum surface structure of such a ball is a.s. determined by its metric measure space structure in a manner which does not depend on the particular quantum surface we are considering, and that a.s. the only isometry from $B_r^\bullet(z; \mathfrak{d}_h)$ to itself is the identity.

In Section 7.3, we use the results of Section 7.2 to reduce Theorem 7.1 to the quantum surface version of the characterization theorem, Theorem 2.4; and we will similarly obtain Theorem 6.2. In Section 7.4, we will prove the aforementioned variant of Theorem 7.2 where we parameterize by quantum natural time.

7.2 Filled metric balls in $\sqrt{8/3}$ -LQG surfaces

In this subsection we will prove some general facts about the filled $\sqrt{8/3}$ -LQG metric balls $B_r^\bullet(z; \mathfrak{d}_h)$ for (D, h) a $\sqrt{8/3}$ -LQG surface. We recall from [MS16b, Corollary 1.5] that $\sqrt{8/3}$ -LQG surfaces are equivalent to Brownian surfaces: in particular, the Brownian map (resp. plane, disk) is equivalent as a metric measure space to the quantum sphere (resp. $\sqrt{8/3}$ -quantum wedge, quantum disk) equipped with its $\sqrt{8/3}$ -LQG metric and area measure. Our first result tells us in particular that there is a.s. only one isometry from a filled metric ball in the Brownian plane to itself.

Proposition 7.3. *Let $(\mathbb{C}, h, 0, \infty)$ be a $\sqrt{8/3}$ -quantum cone and let $B_r^\bullet = B_r^\bullet(0; \mathfrak{d}_h)$ be the filled $\sqrt{8/3}$ -LQG metric ball of radius r centered at 0. Almost surely, the only isometry from $(B_r^\bullet, \mathfrak{d}_{h|_{B_r^\bullet}})$ to itself is the identity.*

Proof. Let \mathcal{I} be the set of isometries from $(B_r^\bullet, \mathfrak{d}_{h|_{B_r^\bullet}})$ to itself. We must show that a.s. \mathcal{I} contains only the identity map. We first observe that a.s. $\phi(0) = 0$ for each $\phi \in \mathcal{I}$ since 0 is a.s. the only point in B_r^\bullet whose distance to every point in ∂B_r^\bullet is equal to r .

Let $\{\Gamma_t\}_{t \geq 0}$ be the metric net process started from 0 and targeted at ∞ , i.e. for each $t \geq 0$,

$$\Gamma_t = \bigcup_{s \leq t} \partial B_s^\bullet.$$

Equivalently, $\{\Gamma_t\}_{t \geq 0}$ is the QLE(8/3, 0) process started from 0 and parameterized by quantum distance time (see [MS15b, Section 6] and [MS16b, Section 2.2]). We also let \mathcal{U}_t be the set of bubbles cut out by Γ_t .

By [MS15a, Proposition 4.4] and local absolute continuity between the Brownian plane and Brownian map (see, e.g., [CL14, Proposition 4]), the law of $\{\Gamma_t\}_{t \in [0, r]}$ (viewed as an increasing family of topological spaces) is absolutely continuous with respect to the law of a $3/2$ -stable Lévy net modulo a deterministic affine change of time (the $3/2$ -stable Lévy net is defined in [MS15a, Section 3]). Furthermore, the conditional law of the bubbles of \mathcal{U}_r given their boundary lengths is absolutely continuous with respect to a collection of independent Brownian disks with a certain measure on areas, which depends only on the boundary lengths.² In particular, it is a.s. the case the following is true.

1. No two distinct bubbles in \mathcal{U}_r are disconnected from ∞ by $\{\Gamma_t\}_{t \in [0, r]}$ at the same time.
2. Each point of Γ_r is a limit of a sequence of sets in \mathcal{U}_r whose diameters tend to 0.

²In fact, these bubbles are exactly Brownian disks if we condition on their areas and boundary lengths: by the analogue of [MS15b, Proposition 6.5] for the $\sqrt{8/3}$ -quantum cone (which is proven in the same manner; c.f. [MS16b, Section 2.2]), the joint law of the quantum surfaces parameterized by the bubbles cut out by $\{\Gamma_t\}_{t \geq 0}$ is the same as the joint law of the quantum surfaces parameterized by the bubbles cut out by a whole-plane SLE₆ from 0 to ∞ independent from h . By [DMS14, Theorem 1.18] these bubbles are independent quantum disks if we condition on their boundary lengths. By [MS16b, Corollary 1.5], quantum disks are equivalent to Brownian disks.

3. The internal metric of \mathfrak{d}_h on each bubble U in \mathcal{U}_r is geodesic and extends continuously to \bar{U} .

Since each map $\phi \in \mathcal{I}$ is an isometry, each such map satisfies $\phi(\Gamma_t) = \Gamma_t$ for each $t \in [0, r]$. In particular, for each $t \in [0, r]$ the image of each bubble in \mathcal{U}_t under ϕ must be another bubble in \mathcal{U}_t . By property 1 above, it follows that a.s. $\phi(U) = U$ for each $U \in \mathcal{U}_r$ and each $\phi \in \mathcal{I}$. By combining this with property 2, we infer that a.s. $\phi(z) = z$ for each $z \in \Gamma_r$ and each $\phi \in \mathcal{I}$. In particular, ϕ fixes the boundary of each $U \in \mathcal{U}_r$ pointwise. Since each $\phi \in \mathcal{I}$ is an isometry, each such map restricts to an isometry from U to U with respect to the internal metric on U for each $U \in \mathcal{U}_r$. By [MS15a, Proposition 2.3] and property 3 above, a.s. $\phi|_U$ is the identity map for each $U \in \mathcal{U}_r$ and each $\phi \in \mathcal{I}$. Therefore a.s. each $\phi \in \mathcal{I}$ is the identity map. \square

We next prove that the quantum surface structure of a filled metric ball a.s. determines and is determined by its metric measure space structure. This result will be deduced from [MS16c, Theorem 1.4] (which gives the analogous statement for a full quantum sphere, plane, or disk).

Proposition 7.4. *For each $r > 0$, there exists a deterministic function F_r from the set of pointed metric measure spaces to the set of singly marked quantum surfaces and a deterministic function G_r going in the opposite direction such that the following is true. Let (D, h) be a quantum surface with the following property. For each bounded open set $V \subset D$ lying at positive distance from ∂D and each $R > 0$ such that $\bar{V} \subset B_R(0)$, the field $h|_V$ is absolutely continuous with respect to the corresponding restriction of a whole-plane GFF normalized so that its circle average over $\partial B_R(0)$ is zero. Let U be any open subset of D such that $\mu_h(U) < \infty$ a.s. and let z be sampled uniformly from $\mu_h|_U$, normalized to be a probability measure. On the event $\{\mathfrak{d}_h(z, \partial U) > r\}$, a.s.*

$$\begin{aligned} F_r\left(B_r^\bullet(z; \mathfrak{d}_h), \mathfrak{d}_h|_{B_r^\bullet(z; \mathfrak{d}_h)}, \mu_h|_{B_r^\bullet(z; \mathfrak{d}_h)}, z\right) &= \left(B_r^\bullet(z; \mathfrak{d}_h), h|_{B_r^\bullet(z; \mathfrak{d}_h)}, z\right) \quad \text{and} \\ G_r\left(B_r^\bullet(z; \mathfrak{d}_h), h|_{B_r^\bullet(z; \mathfrak{d}_h)}, z\right) &= \left(B_r^\bullet(z; \mathfrak{d}_h), \mathfrak{d}_h|_{B_r^\bullet(z; \mathfrak{d}_h)}, \mu_h|_{B_r^\bullet(z; \mathfrak{d}_h)}, z\right). \end{aligned} \quad (7.1)$$

We emphasize that the functions F_r and G_r appearing in Proposition 7.4 do *not* depend on the particular law of the quantum surface (D, h) .

We will deduce Proposition 7.4 from a similar but less general statement which applies only in the case of a $\sqrt{8/3}$ -quantum cone, together with an absolute continuity argument. In the next several lemmas, we let $(\mathbb{C}, h, 0, \infty)$ be a $\sqrt{8/3}$ -quantum cone and to lighten notation we write

$$B_r^\bullet := B_r^\bullet(0; \mathfrak{d}_h).$$

The main input in the proof of Proposition 7.4 is the following lemma.

Lemma 7.5. *For each $r > 0$, the pointed metric measure space $\left(B_r^\bullet, \mathfrak{d}_h|_{B_r^\bullet}, \mu_h|_{B_r^\bullet}, 0\right)$ and the quantum surface $(B_r^\bullet, h|_{B_r^\bullet}, 0)$ a.s. determine each other. In other words, there exist deterministic functions F_r and G_r such that (7.1) holds a.s. with $(\mathbb{C}, h, 0, \infty)$ a γ -quantum cone and $z = 0$.*

For the proof of Lemma 7.5, we first need to collect some facts about filled metric balls which follow from results in [MS15b, MS16b, MS16c]. Our first lemma will be used in particular to relate $h|_{B_r^\bullet}$ to the internal metric of \mathfrak{d}_h on B_r^\bullet .

Lemma 7.6. *Almost surely, for each open set $U \subset \mathbb{C}$ the $\sqrt{8/3}$ -LQG metric $\mathfrak{d}_h|_U$ is equal to the internal metric of \mathfrak{d}_h on U .*

Proof. This is essentially obvious from the construction of the $\sqrt{8/3}$ -LQG metric, but we give a careful justification for the sake of completeness. Let \mathcal{V} be the set of those open sets $V \subset \mathbb{C}$ which can be expressed as a finite union of open Euclidean balls with rational radii whose center has rational coordinates. By [GM16b, Lemma 2.2], it is a.s. the case that for each $V \in \mathcal{V}$, the $\sqrt{8/3}$ -LQG metric $\mathfrak{d}_h|_V$ induced by $h|_V$ is equal to the internal metric of h on V .

Suppose now that this is the case (which happens with probability 1) and fix an open set $U \subset \mathbb{C}$. Almost surely, there is an increasing sequence $\{V_n\}_{n \in \mathbb{N}}$ of open sets belonging to \mathcal{V} whose closures are contained in U and whose union is U . By definition (c.f. [GM16b, Section 2.2.2]), for each $z, w \in U$,

$$\mathfrak{d}_h|_U(w_1, w_2) = \lim_{n \rightarrow \infty} \mathfrak{d}_h|_{V_n}(z, w).$$

Each path from z to w contained in U is contained in V_n large enough $n \in \mathbb{N}$, in which case its $\mathfrak{d}_h|_{V_n}$ -length is the same as its $\mathfrak{d}_h|_U$ -length and its \mathfrak{d}_h -length. The statement of the lemma follows. \square

We next collect some facts about the laws of the quantum surfaces parameterized by the filled metric ball B_r^\bullet and its complement.

Lemma 7.7. *Let $r > 0$ and let w be sampled uniformly from $\nu_h|_{\partial B_r^\bullet}$, normalized to be a probability measure. The quantum surfaces*

$$(B_r^\bullet, h|_{B_r^\bullet}, 0, w) \quad \text{and} \quad (\mathbb{C} \setminus B_r^\bullet, h|_{\mathbb{C} \setminus B_r^\bullet}, \infty, w)$$

parameterized by a filled metric ball and its complement are conditionally independent given the quantum length $\nu_h(\partial B_r^\bullet)$.

Proof. This follows from the construction of the metric \mathfrak{d}_h via QLE(8/3, 0) in [MS15b, MS16b]. \square

Lemma 7.8. *Let $r > 0$ and let w be sampled uniformly from $\nu_h|_{\partial B_r^\bullet}$, normalized to be a probability measure. Almost surely, $(\mathbb{C}, \mu_h, \mathfrak{d}_h, 0)$ is the metric space quotient of $(B_r^\bullet, \mathfrak{d}_h|_{B_r^\bullet}, \mu_h|_{B_r^\bullet}, 0, w)$ and $(\mathbb{C} \setminus B_r^\bullet, \mathfrak{d}_h|_{\mathbb{C} \setminus B_r^\bullet}, \mu_h|_{\mathbb{C} \setminus B_r^\bullet}, \infty, w)$, glued together according to quantum length along their boundaries in such a way that the second marked points in the two metric measure spaces are identified. In particular, $(\mathbb{C}, \mu_h, \mathfrak{d}_h, 0, w)$ is a.s. determined by $(B_r^\bullet, \mathfrak{d}_h|_{B_r^\bullet}, \mu_h|_{B_r^\bullet}, 0, w)$ and $(\mathbb{C} \setminus B_r^\bullet, \mathfrak{d}_h|_{\mathbb{C} \setminus B_r^\bullet}, \mu_h|_{\mathbb{C} \setminus B_r^\bullet}, \infty, w)$.*

Proof. The analogous statement in the case when $(\mathbb{C}, h, 0, \infty)$ is a doubly marked unit area quantum sphere rather than a γ -quantum cone, in which case $(\mathbb{C}, \mu_h, \mathfrak{d}_h, 0)$ is equivalent to the Brownian map with two marked points sampled uniformly from its area measure, is explained just after the statement of [MS15a, Proposition 2.7]. The case of a γ -quantum cone follows since the γ -quantum cone is equivalent to the Brownian plane and the local behavior of the latter at its marked point is the same as the local behavior of the Brownian map at a uniformly random point sampled from its area measure [CL14, Proposition 4]. \square

Proof of Lemma 7.5. To lighten notation, define the quantum surface and metric measure space

$$\mathcal{S}_r^\bullet := (B_r^\bullet, h|_{B_r^\bullet}, 0, w) \quad \text{and} \quad \mathfrak{B}_r^\bullet := (B_r^\bullet, \mathfrak{d}_h|_{B_r^\bullet}, \mu_h|_{B_r^\bullet}, 0, w). \quad (7.2)$$

Since $h|_{B_r^\bullet}$ a.s. determines $\mathfrak{d}_h|_{B_r^\bullet}$ and $\mu_h|_{B_r^\bullet}$ by construction, it is clear that \mathcal{S}_r^\bullet a.s. determines \mathfrak{B}_r^\bullet .

We now prove the reverse is true. In fact, in order to apply Lemma 7.8, we will first prove a slightly different statement. Namely, let w be sampled uniformly from $\nu_h|_{\partial B_r^\bullet}$ and define the quantum surface $\mathcal{S}_r^\bullet(w)$ and the metric measure space $\mathfrak{B}_r^\bullet(w)$ as in (7.2) but with the extra marked boundary point w . We will show that the metric measure space $\mathfrak{B}_r^\bullet(w)$ a.s. determines the quantum surface $\mathcal{S}_r^\bullet(w)$.

By [MS16c, Theorem 1.4], the metric measure space $(\mathbb{C}, \mathfrak{d}_h, \mu_h, 0, w)$ (which is a Brownian plane with an extra marked point) and the quantum surface $(\mathbb{C}, h, 0, \infty, w)$ a.s. determine each other. In particular, $\mathcal{S}_r^\bullet(w)$ is a.s. determined by $(\mathbb{C}, \mathfrak{d}_h, \mu_h, 0, w)$. To complete the proof it suffices to show that $\mathcal{S}_r^\bullet(w)$ is conditionally independent from $(\mathbb{C}, \mathfrak{d}_h, \mu_h, 0, w)$ given $\mathfrak{B}_r^\bullet(w)$.

By Lemma 7.7, the interior and exterior quantum surfaces $\mathcal{S}_r^\bullet(w)$ and $(\mathbb{C} \setminus B_r^\bullet, h|_{\mathbb{C} \setminus B_r^\bullet}, \infty, w)$ are conditionally independent given $\nu_h(\partial B_r^\bullet)$. In particular, the exterior metric measure space $(\mathbb{C} \setminus B_r^\bullet, \mathfrak{d}_h|_{\mathbb{C} \setminus B_r^\bullet}, \mu_h|_{\mathbb{C} \setminus B_r^\bullet}, \infty, w)$ is conditionally independent from $\mathcal{S}_r^\bullet(w)$ given $\nu_h(\partial B_r^\bullet)$.

By Lemma 7.8, the interior and exterior metric measure spaces $\mathfrak{B}_r^\bullet(w)$ and $(\mathbb{C} \setminus B_r^\bullet, \mathfrak{d}_h|_{\mathbb{C} \setminus B_r^\bullet}, \mu_h|_{\mathbb{C} \setminus B_r^\bullet}, \infty, w)$ together a.s. determine the full metric measure space $(\mathbb{C}, \mathfrak{d}_h, \mu_h, 0, w)$. The discussion in [MS15a, Section 3.6] (together with local absolute continuity between the Brownian plane and Brownian map) implies that $\nu_h(\partial B_r^\bullet)$ is a.s. determined by $\mathfrak{B}_r^\bullet(w)$. Hence $(\mathbb{C}, \mathfrak{d}_h, \mu_h, 0, w)$ is conditionally independent from $\mathcal{S}_r^\bullet(w)$ given $\mathfrak{B}_r^\bullet(w)$. Therefore, $\mathfrak{B}_r^\bullet(w)$ a.s. determines $\mathcal{S}_r^\bullet(w)$.

The conditional law of $\mathfrak{B}_r^\bullet(w)$ given $(\mathbb{C}, \mathfrak{d}_h, \mu_h, 0)$ depends only on \mathfrak{B}_r^\bullet (since w is just a uniform sample from the boundary measure on \mathfrak{B}_r^\bullet). In particular, $\mathfrak{B}_r^\bullet(w)$ and \mathcal{S}_r^\bullet (which is a.s. determined by $(\mathbb{C}, \mathfrak{d}_h, \mu_h, 0)$) are conditionally independent given \mathfrak{B}_r^\bullet , so \mathfrak{B}_r^\bullet a.s. determines \mathcal{S}_r^\bullet . \square

Proposition 7.4 concerns a filled metric ball centered at a uniformly random point, whereas Lemma 7.5 concerns a filled metric ball centered at the fixed point 0. The following lemma will be used to bridge the gap between these choices of center point.

Lemma 7.9. *Let h be any embedding of a γ -quantum cone into $(\mathbb{C}, 0, \infty)$. Let $V \subset \mathbb{C}$ be a deterministic bounded open set and let z be sampled uniformly from $\mu_h|_V$, normalized to be a probability measure. Then the law of the quantum surface $(\mathbb{C}, h, z, \infty)$ is absolutely continuous with respect to the law of $(\mathbb{C}, h, 0, \infty)$.*

Proof. Let η' be a whole-plane space-filling $\text{SLE}_{\kappa'}$ from ∞ to ∞ , sampled independently from h and then parameterized by γ -quantum mass with respect to h in such a way that $\eta'(0) = 0$. Let t be sampled uniformly from $[0, 1]$. If A is a bounded subset of \mathbb{C} with $\mu_h(A) > 0$, then $\mathbb{P}[A \subset \eta'([0, 1]) | h] > 0$ and hence $\mathbb{P}[\eta'(t) \in A | h] > 0$. Therefore, the conditional law of z given h is a.s. absolutely continuous with respect to the conditional law of $\eta'(t)$ given h . Hence the joint law of (h, z) is absolutely continuous with respect to the joint law of $(h, \eta'(t))$. By [DMS14, Lemma 9.3], $(\mathbb{C}, h, \eta'(t), \infty) \stackrel{d}{=} (\mathbb{C}, h, 0, \infty)$. The conclusion of the lemma follows. \square

Proof of Proposition 7.4. Fix $r > 0$ and $\epsilon \in (0, 1)$. Let (D, h) be a quantum surface as in the proposition statement, let U be an open subset of D such that $\mu_h(U) < \infty$, and let z be sampled uniformly from $\mu_h|_U$, normalized to be a probability measure. On the event $\{\mathfrak{d}_h(z, \partial U) > r\}$, the filled metric ball $B_r^\bullet(z; \mathfrak{d}_h)$ a.s. lies at positive Euclidean distance from ∂U . Hence there exists a bounded open set $V \subset U$ such that

$$\mathbb{P}[B_r^\bullet(z; \mathfrak{d}_h) \subset V, \mathfrak{d}_h(z, \partial U) > r] \geq (1 - \epsilon)\mathbb{P}[\mathfrak{d}_h(z, \partial U) > r]. \quad (7.3)$$

By Lemma 7.6, on the event $\{B_r^\bullet(z; \mathfrak{d}_h) \subset V\}$, we have $B_r^\bullet(z; \mathfrak{d}_h) = B_r^\bullet(z; \mathfrak{d}_{h|_V})$. By assumption, the law of the field $h|_V$ is absolutely continuous with respect to the corresponding restriction of a whole-plane GFF normalized so that its circle average over $\partial B_R(0)$ is 0, for $R > 0$ such that $\bar{V} \subset B_R(0)$. This law, in turn, is absolutely continuous with respect to the law of $\hat{h}|_V$, where \hat{h} is an appropriate choice of embedding of a $\sqrt{8/3}$ -quantum cone into \mathbb{C} .

By Lemma 7.9, if w is sampled uniformly from $\mu_{\hat{h}}|_V$, normalized to be a probability measure, then the law of the quantum surface $(\mathbb{C}, \hat{h}, w, \infty)$ is absolutely continuous with respect to the law of a $\sqrt{8/3}$ -quantum cone. By definition, the metrics $\mathfrak{d}_{h|_V}$ and $\mathfrak{d}_{\hat{h}|_V}$ are obtained by applying the same deterministic functional to h and \hat{h} . Hence the law of the quantum surface $(B_r^\bullet(z; \mathfrak{d}_h), h|_{B_r^\bullet(z; \mathfrak{d}_h)}, z)$ is absolutely continuous with respect to the law of $(B_r^\bullet(0; \mathfrak{d}_{\hat{h}}), \hat{h}|_{B_r^\bullet(0; \mathfrak{d}_{\hat{h}})}, 0)$ on the event $\{B_r^\bullet(z; \mathfrak{d}_h) \subset V\}$. By Lemma 7.5, we infer that (7.1) occurs a.s. on this event. By (7.3) and since $\epsilon \in (0, 1)$ is arbitrary, we infer that (7.1) holds a.s. \square

7.3 Proof of Theorems 7.1 and 7.2

Suppose we are in the setting of Theorem 7.1. We will prove the theorem by reducing it to Theorem 2.4. Let $(\mathbb{C}, h, 0, \infty)$ be the γ -quantum cone whose associated $\sqrt{8/3}$ -LQG metric measure space structure is given by $(\tilde{X}, \tilde{d}, \tilde{\mu}, \tilde{x})$, which a.s. exists and is unique by [MS16c, Theorem 1.4]. In other words, if $\mathfrak{d}_{\tilde{h}}$ and $\mu_{\tilde{h}}$ are the $\sqrt{8/3}$ -LQG metric and measure induced by \tilde{h} , respectively, then $(\mathbb{C}, \mathfrak{d}_{\tilde{h}}, \mu_{\tilde{h}}, 0)$ and $(\tilde{X}, \tilde{d}, \tilde{\mu}, \tilde{x})$ are equivalent as pointed metric measure spaces.

Fix some choice of embedding \tilde{h} and henceforth assume without loss of generality that in fact $(\mathbb{C}, \mathfrak{d}_{\tilde{h}}, \mu_{\tilde{h}}, 0) = (\tilde{X}, \tilde{d}, \tilde{\mu}, \tilde{x})$, so that $\tilde{\eta}' : \mathbb{R} \rightarrow \mathbb{C}$ and $\tilde{\eta}'(0) = 0$; and the homeomorphism Φ of condition 2 takes \mathbb{C} to \mathbb{C} . We will check that the pair $(\tilde{h}, \tilde{\eta}')$ satisfies the conditions of Theorem 2.4. The following lemma implies in particular the statement about the law of the future beaded quantum surfaces in condition 1 of Theorem 2.4.

Lemma 7.10. *For $t \in \mathbb{R}$, let $\tilde{\mathcal{S}}_{t, \infty} = (\tilde{\eta}'([t, \infty)), \tilde{h}|_{\tilde{\eta}'([t, \infty))}, \tilde{\eta}'(t), \infty)$ be the future quantum surface defined as in Theorem 2.4 for the above choice of $(\tilde{h}, \tilde{\eta}')$. Then $\tilde{\mathcal{S}}_{t, \infty}$ has the law of a $\sqrt{6}$ -quantum wedge. Furthermore, the ordered collection of connected components of the metric measure space $(\tilde{X}_{t, \infty}, \tilde{d}_{t, \infty}, \tilde{\mu}|_{\tilde{X}_{t, \infty}})$ from Theorem 7.1 a.s. determines the quantum surface $\tilde{\mathcal{S}}_{t, \infty}$.*

Proof. Fix $t \in \mathbb{R}$ and for $s \geq t$, let \tilde{U}_t^s be the connected component of the interior of $\eta'([t, \infty))$ containing s (or let $\tilde{U}_t^s = \{\eta'(s)\}$ if $\eta'(s) \in \partial\eta'([t, \infty))$). Also let \tilde{x}_t^s (resp. \tilde{y}_t^s) be the point of $\partial\tilde{U}_t^s$ where $\tilde{\eta}'$ starts (resp. finishes) filling in \tilde{U}_t^s . By Lemma 7.6, it is a.s. the case that the internal metric of $\mathfrak{d}_{\tilde{h}}$ on each \tilde{U}_t^s coincides with the $\sqrt{8/3}$ -LQG metric $\mathfrak{d}_{\tilde{h}|_{\tilde{U}_t^s}}$.

By condition 1 in Theorem 7.1, the collection of doubly pointed metric measure spaces $\{(\tilde{U}_t^s, \mathfrak{d}_{\tilde{h}|_{\tilde{U}_t^s}}, \mu_{h|_{\tilde{U}_t^s}}, \tilde{x}_t^s, \tilde{y}_t^s)\}_{s \geq t}$ agrees in law with the collection of beads of a $\sqrt{6}$ -quantum wedge, parameterized by the accumulated quantum mass of the previous beads. By [MS16c, Theorem 1.4], a $\sqrt{6}$ -quantum wedge is a.s. determined by its metric measure space structure. Let $\mathring{\mathcal{S}}_{t,\infty}^s$ be the $\sqrt{6}$ -quantum wedge determined by $\{(\tilde{U}_t^s, \mathfrak{d}_{\tilde{h}|_{\tilde{U}_t^s}}, \mu_{h|_{\tilde{U}_t^s}}, \tilde{x}_t^s, \tilde{y}_t^s)\}_{s \geq t}$.

For $s \geq t$, let $\mathring{\mathcal{D}}_t^s$ be the first bead of $\mathring{\mathcal{S}}_{t,\infty}^s$ with the property that the sum of the quantum areas of the previous beads is at least $s - t$, so that $\mathring{\mathcal{D}}_t^s$ is a doubly marked quantum surface. Let \mathring{h}_t^s be an embedding of $\mathring{\mathcal{D}}_t^s$ into $(\mathbb{D}, 1, -1)$ (chosen in some manner which depends only on $\mathring{\mathcal{S}}_{t,\infty}^s$). Then for each rational $s \geq t$, there a.s. exists an isometry $f_t^s : (\mathbb{D}, \mathfrak{d}_{\mathring{h}_t^s}) \rightarrow (\tilde{U}_t^s, \mathfrak{d}_{\tilde{h}|_{\tilde{U}_t^s}})$ such that $(f_t^s)_* \mu_{\mathring{h}_t^s} = \mu_{\tilde{h}|_{\tilde{U}_t^s}}$, $f_t^s(1) = \tilde{x}_t^s$, and $f_t^s(-1) = \tilde{y}_t^s$.

Now fix $s \geq t$. To prove the first statement of the lemma, we will show that the map f_t^s is a.s. conformal, so that $(\mathbb{D}, \mathring{h}_t^s, 1, -1)$ and $(\tilde{U}_t^s, \tilde{h}|_{\tilde{U}_t^s}, \tilde{x}_t^s, \tilde{y}_t^s)$ are equivalent as quantum surfaces. To this end, let $z \in \mathbb{D}$ be sampled uniformly from $\mu_{\mathring{h}_t^s}$, normalized to be a probability measure. For $r > 0$, define the metric balls

$$\mathring{B}_r^\bullet := B_r^\bullet(z; \mathfrak{d}_{\mathring{h}_t^s}) \quad \text{and} \quad \tilde{B}_r^\bullet := B_r^\bullet(f_t^s(z); \mathfrak{d}_{\tilde{h}}),$$

so that (since f_t^s is an isometry) $f_t^s(\mathring{B}_r^\bullet) = \tilde{B}_r^\bullet$. Also define the event

$$E_t^s(r) := \{\mathfrak{d}_{\mathring{h}_t^s}(z, \partial\mathbb{D}) > r\} = \{\mathfrak{d}_{\tilde{h}}(f_t^s(z), \partial\tilde{U}_t^s) > r\}, \quad (7.4)$$

and note that the two definitions agree since f_t^s is an isometry.

By Proposition 7.4, if we let F_r be the deterministic function from that proposition then on the event $E_t^s(r)$, it is a.s. the case that

$$F_r\left(\mathring{B}_r^\bullet, \mathfrak{d}_{\mathring{h}_t^s|_{\mathring{B}_r^\bullet}}, \mu_{\mathring{h}_t^s|_{\mathring{B}_r^\bullet}}, z\right) = \left(\mathring{B}_r^\bullet, \mathring{h}_t^s|_{\mathring{B}_r^\bullet}, z\right). \quad (7.5)$$

Since f_t^s is measure-preserving, the point $f_t^s(z)$ is sampled uniformly from $\mu_{\tilde{h}|_{\tilde{U}_t^s}}$, normalized to be a probability measure. Since \tilde{U}_t^s is a.s. bounded, if we are given $\epsilon \in (0, 1)$, we can find a deterministic bounded open set $V \subset \mathbb{C}$ such that

$$\mathbb{P}[\tilde{U}_t^s \subset V] \geq 1 - \epsilon.$$

On the event $\{\tilde{U}_t^s \subset V\}$, the conditional law of z given \tilde{h} is a.s. absolutely continuous with respect to a uniform sample from $\mu_{\tilde{h}|_V}$, normalized to be a probability measure. Since $\epsilon \in (0, 1)$ is arbitrary and \tilde{h} is an embedding of a $\sqrt{8/3}$ -quantum cone, it follows from Proposition 7.4 that on the event $E_t^s(r)$, a.s.

$$F_r\left(\tilde{B}_r^\bullet, \mathfrak{d}_{\tilde{h}|_{\tilde{B}_r^\bullet}}, \mu_{\tilde{h}|_{\tilde{B}_r^\bullet}}, f_t^s(z)\right) = \left(\tilde{B}_r^\bullet, \tilde{h}|_{\tilde{B}_r^\bullet}, f_t^s(z)\right). \quad (7.6)$$

Since f_t^s is a measure-preserving isometry, on the event $E_t^s(r)$ the left sides of (7.5) and (7.6) a.s. coincide, so we have the equality of quantum surfaces

$$\left(\mathring{B}_r^\bullet, \mathring{h}_t^s|_{\mathring{B}_r^\bullet}, z\right) = \left(\tilde{B}_r^\bullet, \tilde{h}|_{\tilde{B}_r^\bullet}, f_t^s(z)\right), \quad \text{on } E_t^s(r). \quad (7.7)$$

Hence on this event there a.s. exists a conformal map $g_z : \mathring{B}_r^\bullet \rightarrow \tilde{B}_r^\bullet$ such that

$$\tilde{h}|_{\tilde{B}_r^\bullet} = \mathring{h}_t^s \circ g_z^{-1} + Q \log |(g_z^{-1})'|.$$

By the LQG coordinate change formula for \mathfrak{d}_h (recall Section 2.2.4), a.s. g_z is an isometry from $\left(\mathring{B}_r^\bullet, \mathfrak{d}_{\mathring{h}_t^s|_{\mathring{B}_r^\bullet}}\right)$ to $\left(\tilde{B}_r^\bullet, \mathfrak{d}_{\tilde{h}|_{\tilde{B}_r^\bullet}}\right)$. Since f_t^s also restricts to an isometry between these metric spaces, it follows from Proposition 7.3 (applied to the isometry $(f_t^s)^{-1} \circ g_z$) and local absolute continuity that a.s. $g_z = f_t^s|_{\mathring{B}_r^\bullet}$. In other words, f_t^s is a.s. conformal on \mathring{B}_r^\bullet on the event $E_t^s(r)$.

Since z was a uniform sample from the measure $\mu_{h_t^s}$, whose closed support is a.s. equal to all of \mathbb{D} , for each $w \in \mathbb{D}$ it a.s. holds with positive conditional probability given $(\tilde{h}, \tilde{\eta}')$, $\tilde{\mathcal{S}}_{t,\infty}$, and f_t^s that $\mathfrak{d}_{\tilde{h}_t^s}(z, w) \leq \frac{1}{2}\mathfrak{d}_{\tilde{h}_t^s}(z, \partial\mathbb{D})$. If this is the case then the above discussion implies that f_t^s is conformal on a neighborhood of w and that the restriction of h_t^s (resp. \tilde{h}) to this neighborhood (resp. its image under f_t^s) are related by the $\sqrt{8/3}$ -LQG coordinate change formula. Hence f_t^s is a.s. conformal on all of \mathbb{D} and $\tilde{h}|_{\tilde{U}_t^s} = \dot{h}_t^s \circ (f_t^s)^{-1} + Q \log |(f_t^s)^{-1}|'$. Therefore, a.s. $\tilde{\mathcal{S}}_{t,\infty} = \dot{\mathcal{S}}_{t,\infty}$ as quantum surfaces.

By construction $\dot{\mathcal{S}}_{t,\infty}$ is a.s. determined by $(\tilde{X}_{t,\infty}, \tilde{d}_{t,\infty}, \tilde{\mu}|_{\tilde{X}_{t,\infty}}, \tilde{\eta}'(t))$ and we showed above that $\dot{\mathcal{S}}_{t,\infty} = \tilde{\mathcal{S}}_{t,\infty}$ a.s., which yields the last statement of the lemma. \square

We next check the independence conditions in Theorem 2.4. In light of Lemma 7.10, it suffices to prove that if $\tilde{\mathcal{F}}_{-\infty,t}^m$ and $\tilde{\mathcal{F}}_{t,\infty}^m$ are the σ -algebras from Theorem 7.1, then $\tilde{\mathcal{F}}_{-\infty,t}^m$ (resp. $\tilde{\mathcal{F}}_{t,\infty}^{c,m}$) contains the σ -algebra generated by the quantum surfaces $\tilde{\mathcal{S}}_{v_Z(s),s} = (\tilde{\eta}'([v_Z(s), s]), \tilde{h}|_{\tilde{\eta}'([v_Z(s), s])}, \tilde{\eta}'(s))$ as s ranges over all $\pi/2$ -cone intervals for Z which are maximal in some interval in $(-\infty, t)$ (resp. (t, ∞)) with rational endpoints. By Lemma 7.6, in the notation of Theorem 7.1 it holds that

$$\left(\tilde{X}_{v_Z(s),s}, \tilde{d}_{v_Z(s),s}, \tilde{\mu}|_{\tilde{X}_{v_Z(s),s}}, \tilde{\eta}'_{\tilde{X}_{v_Z(s),s}}(s) \right) = \left(\tilde{\eta}'([v_Z(s), s]), \mathfrak{d}_{\tilde{h}|_{\tilde{\eta}'([v_Z(s), s])}}, \mu_{\tilde{h}|_{\tilde{\eta}'([v_Z(s), s])}}, \tilde{\eta}'(s) \right)$$

as pointed metric measure spaces. It therefore suffices to prove the following.

Lemma 7.11. *Suppose we are in the setting of Theorem 2.4. Let $[a, b] \subset \mathbb{R}$ be an interval with rational endpoints. Also let $c \in (a, b)$ and let s be the maximal $\pi/2$ -cone time for Z in $[a, b]$ with $c \in [v_Z(s), s]$. The quantum surface $\tilde{\mathcal{S}}_{v_Z(s),s} = (\tilde{\eta}'([v_Z(s), s]), \tilde{h}|_{\tilde{\eta}'([v_Z(s), s])}, \tilde{\eta}'(s))$ is a.s. determined by its corresponding metric measure space structure $(\tilde{\eta}'([v_Z(s), s]), \mathfrak{d}_{\tilde{h}|_{\tilde{\eta}'([v_Z(s), s])}}, \mu_{\tilde{h}|_{\tilde{\eta}'([v_Z(s), s])}}, \tilde{\eta}'(s))$.*

Proof. To lighten notation, let \tilde{U}_s be the interior of the bubble $\tilde{\eta}'([v_Z(s), s])$. Condition on the metric measure space $(\tilde{U}_s, \mathfrak{d}_{\tilde{h}|_{\tilde{U}_s}}, \mu_{\tilde{h}|_{\tilde{U}_s}}, \tilde{\eta}'(s))$ and let $\tilde{\mathcal{S}}'_{v_Z(s),s}$ be a singly marked quantum surface sampled from the conditional law of $\tilde{\mathcal{S}}_{v_Z(s),s}$ given $(\tilde{U}_s, \mathfrak{d}_{\tilde{h}|_{\tilde{U}_s}}, \mu_{\tilde{h}|_{\tilde{U}_s}}, \tilde{\eta}'(s))$ in such a way that it is conditionally independent from $\tilde{\mathcal{S}}_{v_Z(s),s}$. Then $\tilde{\mathcal{S}}_{v_Z(s),s}$ and $\tilde{\mathcal{S}}'_{v_Z(s),s}$ are two quantum surfaces with the same law which a.s. generate the same metric measure space structure. We must show that a.s. $\tilde{\mathcal{S}}_{v_Z(s),s} = \tilde{\mathcal{S}}'_{v_Z(s),s}$ as quantum surfaces (we note that this is not immediate from the results of [MS16c] since we do not a priori know the exact law of these quantum surfaces).

Let \tilde{h}'_s be an embedding of $\tilde{\mathcal{S}}'_{v_Z(s),s}$ into $(\mathbb{D}, 1)$. By our choice of coupling, there a.s. exists an isometry $f : (\mathbb{D}, \tilde{h}'_s) \rightarrow (\tilde{U}_s, \mathfrak{d}_{\tilde{h}|_{\tilde{U}_s}})$ such that $f_*\mu_{\tilde{h}'_s} = \mu_{\tilde{h}|_{\tilde{U}_s}}$ and $f(1) = \tilde{\eta}'(s)$.

If we sample z uniformly from $\mu_{\tilde{h}_s}$, then by Propositions 7.3 and 7.4 and an absolute continuity argument as in the proof of Lemma 7.10 (here we use that $\tilde{\mathcal{S}}_{v_Z(s),s}$ is a sub-surface of a $\sqrt{8/3}$ -quantum cone and that $\tilde{\mathcal{S}}'_{v_Z(s),s} \stackrel{d}{=} \tilde{\mathcal{S}}_{v_Z(s),s}$), we find that for $r > 0$, a.s.

$$\left(B_r^\bullet(z; \mathfrak{d}_{\tilde{h}}), \tilde{h}|_{B_r^\bullet(z; \mathfrak{d}_{\tilde{h}})} \right) = \left(B_r^\bullet(z; \mathfrak{d}_{\tilde{h}'_s}), \tilde{h}'_s|_{B_r^\bullet(z; \mathfrak{d}_{\tilde{h}'_s})} \right)$$

as quantum surfaces on the event $\{\mathfrak{d}_{\tilde{h}'_s}(z, \partial\mathbb{D}) > r\}$. By the same argument used in the proof of Lemma 7.10, this implies that on the event $\{\mathfrak{d}_{\tilde{h}'_s}(z, \partial\mathbb{D}) > r\}$, f is a.s. conformal on $B_r^\bullet(z; \mathfrak{d}_{\tilde{h}})$ and the restriction of \tilde{h}'_s (resp. \tilde{h}) to $B_r^\bullet(z; \mathfrak{d}_{\tilde{h}'_s})$ (resp. $B_r^\bullet(z; \mathfrak{d}_{\tilde{h}})$) are related by the $\sqrt{8/3}$ -LQG coordinate change formula. Since the closed support of $\mu_{\tilde{h}}$ is a.s. equal to all of \mathbb{C} , it follows that the closed support of $\mu_{\tilde{h}'_s}$ is a.s. equal to all of \mathbb{D} . From this, we infer that f is a.s. conformal on all of \mathbb{D} and that a.s. $\tilde{h}|_{\tilde{U}_s} = \tilde{h} \circ f^{-1} + Q \log |(f^{-1})'|$. Hence a.s. $\tilde{\mathcal{S}}_{v_Z(s),s} = \tilde{\mathcal{S}}'_{v_Z(s),s}$. \square

Proof of Theorem 7.1. As described at the beginning of this subsection, we let $(\mathbb{C}, \tilde{h}, 0, \infty)$ be the $\sqrt{8/3}$ -quantum cone determined by $(\tilde{X}, \tilde{d}, \tilde{\mu}, \tilde{x})$ and view $\tilde{\eta}'$ as a curve in \mathbb{C} . We claim that the pair $(\tilde{h}, \tilde{\eta}')$ satisfies the conditions of Theorem 2.4. Indeed, condition 2 is immediate from the corresponding condition in Theorem 7.1,

and condition 1 in Theorem 2.4 is implied by the corresponding condition in Theorem 7.1 together with Lemmas 7.10 and 7.11. The theorem statement now follows immediately from Theorem 2.4. \square

Proof of Theorem 7.2. This is deduced from Theorem 6.2 and the results of Section 7.2 via essentially the same argument used to prove Theorem 7.1. Note that condition 2 determines the left/right boundary lengths of the doubly marked Brownian disk parameterized by the unbounded connected component of $\tilde{X}^b \setminus \tilde{\eta}^b([0, t])$, even though these boundary lengths are not specified in condition 1. \square

7.4 SLE₆ parameterized by quantum natural time

We now give a version of Theorem 7.2 where we parameterize the curve by quantum natural time and we only condition on the left/right boundary lengths of the Brownian disk, not its area. This is the statement that will be used in [GM17b] to identify the law of a subsequential limit of percolation on random quadrangulations with boundary.

Suppose $(\mathbb{H}, h^b, 0, \infty)$ is a doubly-marked quantum disk and η^b is an independent chordal SLE₆ from 0 to ∞ . Roughly speaking, parameterizing η^b by quantum natural time with respect to h^b is equivalent to parameterizing by the “quantum local time” at the set of times when η^b disconnects a bubble from ∞ . Formally, quantum natural time is defined for a chordal SLE _{κ'} , $\kappa' \in (4, 8)$, and an independent free-boundary GFF on \mathbb{H} with a $\gamma/2$ -log singularity at the origin in [DMS14, Definition 7.14]; and is defined for the pair (h^b, η^b) via local absolute continuity. We will typically denote an SLE₆ curve parameterized by quantum natural time and its associated left/right boundary length process by a superscript q .

Our characterization theorem for SLE₆ parameterized by quantum natural time will be stated in terms of free Boltzmann Brownian disks, which are introduced in [BM15, Section 1.5] and defined as follows. For $\mathfrak{l} > 0$, a *free Boltzmann Brownian disk with boundary length \mathfrak{l}* is the random metric measure space obtained by first sampling a random area \mathfrak{a} from the probability measure $\frac{\mathfrak{l}^3}{\sqrt{2\pi\mathfrak{a}^5}} e^{-\frac{\mathfrak{l}^2}{2\mathfrak{a}}} \mathbb{1}_{(\mathfrak{a} \geq 0)} d\mathfrak{a}$, then sampling a Brownian disk with boundary length \mathfrak{l} and area \mathfrak{a} . Note that a free Boltzmann Brownian disk with boundary length \mathfrak{l} is obtained from a free Boltzmann Brownian disk with unit boundary length by scaling areas by \mathfrak{l}^2 , boundary lengths by \mathfrak{l} , and distances by $\mathfrak{l}^{1/2}$. A singly (resp. doubly) marked free Boltzmann Brownian disk is a free Boltzmann Brownian disk together with two points sampled uniformly from its natural boundary length measure. In the doubly marked case, the left and right boundary lengths are the lengths of the arcs between the two marked points. A free Boltzmann Brownian disk is equivalent (as a metric measure space with a boundary length measure) to a quantum disk with fixed boundary length.

Theorem 7.12 (Chordal SLE₆ characterization with quantum natural time on a Brownian disk). *Let $(\mathfrak{l}^L, \mathfrak{l}^R) \in (0, \infty)^2$ and suppose we are given a coupling of a doubly-marked free Boltzmann Brownian disk $(\tilde{X}^q, \tilde{d}^q, \tilde{\mu}^q, \tilde{x}, \tilde{y})$ with left/right boundary lengths \mathfrak{l}^L and \mathfrak{l}^R , a random continuous curve $\tilde{\eta}^q : [0, \infty) \rightarrow \tilde{X}^q$ from \tilde{x} to \tilde{y} , and a random process $Z^q = (L^q, R^q)$ which has the law of the left/right boundary length process of a chordal SLE₆ on a doubly marked quantum disk with left/right boundary lengths \mathfrak{l}^L and \mathfrak{l}^R , parameterized by quantum natural time (this process is defined at the beginning of Section 6). Assume that the following conditions are satisfied.*

1. (Laws of complementary connected components) *For $u \geq 0$, let $\tilde{\mathcal{U}}_u^q$ be the collection of singly marked metric measure spaces of the form $(U, \tilde{d}_U, \tilde{\mu}_U, \tilde{x}_U)$ where U is a connected component of $\tilde{X}^q \setminus \tilde{\eta}^q([0, u])$, \tilde{d}_U^q is the internal metric of \tilde{d}^q on U , and \tilde{x}_U is the point where $\tilde{\eta}^q$ finishes tracing ∂U . If we condition on $Z^q|_{[0, u]}$, then the conditional law of $\tilde{\mathcal{U}}_u^q$ is that of a collection of independent singly marked free Boltzmann Brownian disks with boundary lengths specified as follows. The elements of $\tilde{\mathcal{U}}_u^q$ corresponding to the connected components of $\tilde{X}^q \setminus \tilde{\eta}^q([0, u])$ which do not have the target point \tilde{y} on their boundaries are in one-to-one correspondence with the downward jumps of the coordinates of $Z^q|_{[0, u]}$, with boundary lengths given by the magnitudes of the corresponding jump. The element of $\tilde{\mathcal{U}}_u^q$ corresponding to the connected component of $\tilde{X}^q \setminus \tilde{\eta}^q([0, u])$ with \tilde{y} on its boundary has boundary length $L_u^q + R_u^q$.*
2. (Topology and consistency) *The topology of $(\tilde{X}^q, \tilde{\eta}^q)$ is determined by Z^q in the same manner as the topology of a chordal SLE₆ on a doubly marked quantum disk, i.e. there is a pair $((\mathbb{H}, h^q, 0, \infty), \eta^q)$ consisting of a doubly marked quantum disk with left/right boundary lengths \mathfrak{l}^L and \mathfrak{l}^R and an independent*

chordal SLE₆ from 0 to ∞ in \mathbb{H} parameterized by quantum natural time and a homeomorphism $\Phi^q : \mathbb{H} \rightarrow \tilde{X}^q$ with $\Phi_*^q \mu_{h^q} = \tilde{\mu}^q$ and $\Phi^q \circ \eta^q = \tilde{\eta}^q$. Moreover, for each $u \in [0, \infty) \cap \mathbb{Q}$, Φ^q a.s. pushes forward the $\sqrt{8/3}$ -quantum length measure with respect to h^q on the boundary of the unbounded connected component of $\mathbb{H} \setminus \eta^q([0, u])$ to the natural boundary length measure on the connected component of $\tilde{X}^q \setminus \tilde{\eta}^q([0, u])$ with $\tilde{\gamma}$ on its boundary.

Then $(\tilde{X}^q, \tilde{d}^q, \tilde{\mu}^q, \tilde{\eta}^q)$ is equivalent (as a curve-decorated metric measure space) to a doubly-marked quantum disk with left/right boundary lengths l_L and l_R equipped with its $\sqrt{8/3}$ -LQG area measure and metric together with an independent chordal SLE₆ between the two marked points, parameterized by quantum natural time.

Note that the process Z^q does not encode the quantum areas of the bubbles disconnected from ∞ by η^q , so we need to assume that the map Φ^q of condition 2 is area-preserving.

In the companion paper [GM17a], we prove that condition 1 is satisfied in the case when $(\tilde{X}^q, \tilde{d}^q, \tilde{\mu}^q, \tilde{\eta}^q)$ is actually a doubly-marked quantum disk equipped with its $\sqrt{8/3}$ -LQG area measure and metric together with an independent chordal SLE₆ between the two marked points, parameterized by quantum natural time; we do not need this statement for the proof of Theorem 7.12, but theorem statement is vacuous without it. We also provide a description of the law of the process Z^q in terms of its Radon-Nikodym derivative with respect to the law of a pair of independent 3/2-stable processes with no upward jumps.

As we will now show, Theorem 7.12 is an easy consequence of Theorem 7.2 (essentially it follows by re-parameterizing the curve $\tilde{\eta}^q$).

Proof of Theorem 7.12. Suppose we are in the setting of Theorem 7.12. Let $\mathfrak{a} = \tilde{\mu}^q(\tilde{X}^q)$ be the random area of the free Boltzmann Brownian disk \tilde{X}^q . For $t \in [0, \mathfrak{a}]$, let σ_t be the smallest $u \geq 0$ for which the μ_{h^q} -mass of the region disconnected from the target point $\tilde{\gamma}$ by $\tilde{\eta}^q([0, u])$ is at least t . Also let $\sigma_t = \sigma_{\mathfrak{a}}$ for $t \geq \mathfrak{a}$. Since the homeomorphism Φ^q in condition 2 satisfies $\Phi_*^q \mu_{h^q} = \tilde{\mu}^q$, σ_t can equivalently be described as the smallest $u \geq 0$ for which the μ_{h^q} -mass of the region disconnected from ∞ by $\eta^q([0, u])$ is at least t .

Let $\tilde{\eta}^b(t) := \tilde{\eta}^q(\sigma_t)$, $\eta^b(t) := \eta^q(\sigma_t)$, and $Z_t^b := Z_{\sigma_t}^q$. Then $\tilde{\eta}^b$ (resp. η^b) is parameterized by the $\mu_{\tilde{h}^b}$ - (resp. μ_{h^b} -) mass it disconnects from ∞ and Z^b is the left/right boundary length process for each of $\tilde{\eta}^b$ and η^b . We will check the hypotheses of Theorem 7.2 for the curve-decorated metric measure space $(\tilde{X}^q, \tilde{d}^q, \tilde{\mu}^q, \tilde{\eta}^b)$, the field-curve pair (h^q, η^b) , and the process Z^b .

It is clear from condition 2 of the present theorem that condition 2 of Theorem 7.2 is satisfied in the present setting, with $\Phi^b = \Phi^q$. Hence we only need to check condition 1 of Theorem 7.2.

For $t \in [0, \mathfrak{a}]$, let $\tau^b(t)$ be the right endpoint of the largest interval of times containing t on which Z^b is constant, as in the discussion just above Theorem 7.2. Equivalently, $\tau^b(t)$ is the $\tilde{\mu}^q$ -mass of the region disconnected from $\tilde{\gamma}$ by $\tilde{\eta}^b([0, t])$. For $u \geq 0$, let $\tilde{\mathcal{G}}_u^q$ be the σ -algebra generated by $Z^q|_{[0, u]}$ and the ordered sequence of $\tilde{\mu}^q$ -masses of the bubbles disconnected from ∞ by $\tilde{\eta}^q$ before time u . Then for $t \geq 0$, σ_t is a $\{\tilde{\mathcal{G}}_u^q\}_{u \geq 0}$ -stopping time and $\tilde{\mathcal{G}}_{\sigma_t}^q$ is the same as the σ -algebra generated by $Z^b|_{[0, t]}$ and $\tau^b(t)$.

For $u \geq 0$, let $\tilde{\mathfrak{W}}_u^q$ be the element of $\tilde{\mathcal{U}}_u^q$ which contains the target point $\tilde{\gamma}$ on its boundary. Now fix $t \geq 0$ and for $n \in \mathbb{N}$ let σ_t^n be the smallest integer multiple of 2^{-n} which is at least σ_t , so that each σ_t^n is a $\{\tilde{\mathcal{G}}_u^q\}_{u \geq 0}$ -stopping time which takes on only countably many possible values and σ_t^n decreases to σ_t . By condition 1 in the theorem statement, for each $n \in \mathbb{N}$ the conditional law given $\tilde{\mathcal{G}}_{\sigma_t^n}^q$ of the collection of metric measure spaces $\tilde{\mathcal{U}}_{\sigma_t^n}^q$ corresponding to the connected components of $\tilde{X}^q \setminus \tilde{\eta}^q([0, \sigma_t^n])$ is as in condition 1 with σ_t^n in place of u , except that we condition on the area of each of each element of $\tilde{\mathcal{U}}_{\sigma_t^n}^q \setminus \{\tilde{\mathfrak{W}}_{\sigma_t^n}^q\}$ in addition to its boundary length.

We now take a limit as $n \rightarrow \infty$ to get the same statement with σ_t in place of σ_t^n . We have $\bigcap_{n=1}^{\infty} (\tilde{\mathcal{U}}_{\sigma_t^n}^q \setminus \{\tilde{\mathfrak{W}}_{\sigma_t^n}^q\}) = \tilde{\mathcal{U}}_{\sigma_t}^q \setminus \{\tilde{\mathfrak{W}}_{\sigma_t}^q\}$ and by continuity of $\tilde{\eta}^q$ a.s. the unexplored metric spaces $\tilde{\mathfrak{W}}_{\sigma_t^n}^q$ converge to $\tilde{\mathfrak{W}}_{\sigma_t}^q$ as $n \rightarrow \infty$ in, e.g., the pointed Gromov-Prokhorov topology [GPW09]. Since Z^q is right continuous, the boundary length $L_{\sigma_t^n}^q + R_{\sigma_t^n}^q$ of $\tilde{\mathfrak{W}}_{\sigma_t^n}^q$ converges to the boundary length $L_{\sigma_t}^q + R_{\sigma_t}^q$ of $\tilde{\mathfrak{W}}_{\sigma_t}^q$ as $n \rightarrow \infty$. By scaling, it is clear that the law of a singly free Boltzmann Brownian disk depends continuously on its boundary length in the Gromov-Prokhorov topology.

Combining the preceding two paragraphs shows that the conditional law given $\tilde{\mathcal{G}}_{\sigma_t}^q$ (equivalently, given $Z^b|_{[0, t]}$ and $\tau^b(t)$) of $\tilde{\mathcal{U}}_{\sigma_t}^q$ is as in condition 1 with σ_t in place of u , except that we condition on the area of each element of $\tilde{\mathcal{U}}_{\sigma_t}^q \setminus \{\tilde{\mathfrak{W}}_{\sigma_t}^q\}$ in addition to its boundary length. Further conditioning on \mathfrak{a} is equivalent to

conditioning on the total area of $\widetilde{\mathfrak{M}}_{\sigma_t}^{\mathfrak{a}}$. Hence condition 1 in Theorem 7.2 holds for the conditional law of $(\widetilde{X}^{\mathfrak{a}}, \widetilde{d}^{\mathfrak{a}}, \widetilde{\mu}^{\mathfrak{a}}, \widetilde{\eta}^{\mathfrak{a}})$, $(h^{\mathfrak{a}}, \eta^{\mathfrak{b}})$ and $Z^{\mathfrak{b}}$ given \mathfrak{a} .

By Theorem 7.2, we infer that $(\widetilde{X}^{\mathfrak{a}}, \widetilde{d}^{\mathfrak{a}}, \widetilde{\mu}^{\mathfrak{a}}, \widetilde{\eta}^{\mathfrak{a}})$ is equivalent (as a curve-decorated metric measure space) to a doubly-marked quantum disk equipped with its $\sqrt{8/3}$ -LQG area measure and metric together with an independent chordal SLE₆ between the two marked points parameterized by the $\sqrt{8/3}$ -quantum mass it disconnects from ∞ . It follows from the definition of quantum natural time that the inverse time change $\sigma^{-1}(u)$ such that $Z_u^{\mathfrak{a}} = Z_{\sigma^{-1}(u)}^{\mathfrak{b}}$ and $\eta^{\mathfrak{a}}(u) = \eta^{\mathfrak{b}}(\sigma^{-1}(u))$ is a deterministic functional of $Z_u^{\mathfrak{a}}$ (σ^{-1} is the inverse of the local time that $Z_u^{\mathfrak{a}}$ at the endpoints of the intervals where it is constant). Since $\widetilde{\eta}^{\mathfrak{a}}(u) = \widetilde{\eta}^{\mathfrak{b}}(\sigma^{-1}(u))$, we obtain the desired description of $(\widetilde{X}^{\mathfrak{a}}, \widetilde{d}^{\mathfrak{a}}, \widetilde{\mu}^{\mathfrak{a}}, \widetilde{\eta}^{\mathfrak{a}})$. \square

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